P 16. $D = \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix},\ z = e^{i\theta} \right\} \cong U(1)$

(a) $d \in D$ $\quad u = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \in SU(2) \quad |\alpha|^2 + |\beta|^2 = 1$

$$u d u^{-1} = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} \begin{pmatrix} \bar{\alpha} & -\beta \\ \bar{\beta} & \alpha \end{pmatrix}$$

$$= \begin{pmatrix} |\alpha|^2 z + |\beta|^2 \bar{z} & \alpha\beta(-z + \bar{z}) \\ \bar{\alpha}\bar{\beta}(-z + \bar{z}) & |\alpha|^2 \bar{z} + |\beta|^2 z \end{pmatrix} \in D$$

$\Rightarrow \alpha = 0$ or $\beta = 0$

$$N_{SU(2)}(D) = \left\{ \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix},\ z \in U(1) \right\} \cup \left\{ \begin{pmatrix} 0 & -\bar{z} \\ z & 0 \end{pmatrix},\ z \in U(1) \right\}$$

$$= D \cup \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} D$$

(b) $$N_{SU(2)}(D)/D = \left\{ \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} D,\ \begin{pmatrix} 0 & -\bar{z} \\ z & 0 \end{pmatrix} D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} D \right\} \cong \mathbb{Z}_2$$

(c) $$\begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} \begin{pmatrix} \bar{\alpha} & 0 \\ 0 & \alpha \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}$$

$$\begin{pmatrix} 0 & -\bar{\alpha} \\ \alpha & 0 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} \begin{pmatrix} 0 & \bar{\alpha} \\ -\alpha & 0 \end{pmatrix} = \begin{pmatrix} \bar{z} & 0 \\ 0 & z \end{pmatrix}$$

(d) should at least contain $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. and some $a = \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix}$. then it contains $a^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

& $a^3 = \begin{pmatrix} 0 & -\bar{z} \\ \bar{z} & 0 \end{pmatrix}$. it's not isomorphic to $\mathbb{Z}_2$.

NB ($N_{SU(2)}(D)/D$ is not a subgroup of $SU(2)$ or $N_{SU(2)}(D)$)

P 17.

G-set X. $\phi : G \longrightarrow S_X$

(a) effective $\Leftrightarrow$ ϕ injective, i.e. $\phi(g) = 1$ iff $g = 1$

$\forall g \neq 1$. $\exists x$ s.t. $g x_1 = x_2 \neq x_1 \Leftrightarrow \forall g \neq 1$. $\phi(g)$ is a nontrivial permutation

$\phi(g) \neq 1$

(b) $\{g_i\}$ are ineffective $\forall g \in G$

$g_i g x = g_i \cdot x' = x'$ $\quad (\forall x \in X)$

$g g_i x = g x = x'$

$\Rightarrow g_i g = g g_i \quad \forall g \in G$

trivial to show $\{g_i\}$ is a group

$\Rightarrow H = \{ g_i : g_i x = x \ \forall x \in X \} \lhd G$

(c) define the action $G/H \times X \longrightarrow X$

$(gH) \cdot x := g x$

$\forall x \in X$, s.t. $(gH) x = x \Leftrightarrow g x = x \Leftrightarrow g \in H \Leftrightarrow gH = H = 1_{G/H}$

P18. X a finite G set.

G - action transitive $\Rightarrow$ one orbit $= X$

Burnside's lemma $\Rightarrow$ $|G| = \sum_{g \in G} |X^g|$

If all g's have fixed points. $\sum_{g \in G} |X^g| \geq \sum_{g \in G} 1 = |G|$

equality holds iff $\forall g$. $|X^g| = 1$

But $|X^e| = |X| > 1$

$\Rightarrow$ $|X^g| = 0$ for some g.

P19

Lemma: G abelian

$p \mid |G|$, p prime $\Rightarrow \exists\, g \in G$. of order p

Proof. $|G| = pm$.

the Lemma holds for $m=1$. since if $|G|=p$.

G is cyclic. as a result of Lagrange theorem

then any element $g \in G$ has order p $(g^p=1)$

Now suppose for a general $m>1$, $\exists\, h \in G$. s.t. h has order t,

i.e. $h^t = 1$

① if $p \mid t$. then $h^{t/p}$ is of order p.

② else $\langle h \rangle$ is a normal subgroup. ($\because$ G is abelian)

$G/\langle h \rangle$ is an abelian group of order

$|G|/t = pm/t$ $\quad (\because |\langle h \rangle| = t)$

then m/t is an integer smaller than m.

by induction. $G/\langle h \rangle$ has an element

of order p

homomorphism $\varphi: G \longrightarrow G/\langle h \rangle$ a surjection.

$g \longmapsto g\langle h \rangle$

if $g\langle h\rangle$ has order p. then

$$\varphi(g^p) = (g\langle h\rangle)^p = 1_{G/\langle h\rangle} = \langle h\rangle$$

$$g^p = h^x \in \langle h\rangle$$

if $h^x = 1 \Rightarrow g^p = 1$

else $\exists y$. s.t. $(h^x)^y = 1 \Rightarrow g^{py} = 1 \Rightarrow (g^y)^p = 1$