P7. Quaternion $\to$ V.

There are many homomorphisms.

One example:

$Q = \{\pm 1, \pm i, \pm j, \pm k\}$ $\quad V = \{1, a, b, ab\}$

$\varphi: Q \to V$

define $\varphi(i) = a$. $\quad \varphi(j) = b \quad \varphi(k) = \varphi(ij) = \varphi(i)\varphi(j) = ab$

$\varphi(-1) = \varphi(i)\varphi(i) = a^2 = 1$

$\varphi(-i) = \varphi(i)\varphi(-1) = a$

$\varphi(-j) = b$, $\quad \varphi(-k) = ab$

$\ker\varphi = \{\pm 1\} \cong \mathbb{Z}_2$

$\mathrm{im}\,\varphi = V$

P8.

$$\begin{tikzcd} \mathbb{Z}_N \arrow[r, "m_{k_1}"] \arrow[d, "\varphi"] & \mathbb{Z}_N \arrow[d, "\varphi"] \\ \mu_N \arrow[r, "\rho_{k_2}"] & \mu_N \end{tikzcd}$$

commutes $\Longleftrightarrow$ $k_1 = k_2 \bmod N$

① $\Leftarrow$ trivial.

② $\Rightarrow$ if $k_1 \neq k_2 \bmod N$

↳ : $\rho_{k_2}(\varphi(\bar{i})) = \rho_{k_2}(\omega^{i+nN}) = \omega^{ik_2 \bmod N}$

⤵ : $\varphi(m_{k_1}(\bar{i})) = \varphi(\overline{k_1 i}) = \omega^{ik_1 \bmod N}$

$\forall i, \quad ik_1 = ik_2 \bmod N$

$k_1 = k_2 \bmod N.$

P 9. $T: \mathbb{C}^2 \to \mathbb{C}^2$ equivariant $\Rightarrow T(\vec{z}) = \alpha \vec{z}$, $\alpha \in \mathbb{C}$

$$\begin{tikzcd}
\mathbb{C}^2 \arrow[r, "T"] \arrow[d, "SU(2)"'] & \mathbb{C}^2 \arrow[d, "SU(2)"] \\
\mathbb{C}^2 \arrow[r] & \mathbb{C}^2
\end{tikzcd}$$

$$M \cdot T(\vec{z}) = T(M \cdot \vec{z}), \quad \forall \vec{z} \in \mathbb{C}^2, \ \forall M \in SU(2) \Rightarrow [T, M] = 0$$

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad M_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad M_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$TM_1 = M_1T \Rightarrow \begin{pmatrix} b & -a \\ d & -c \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} \Rightarrow T = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

$$TM_2 = M_2T \Rightarrow \begin{pmatrix} bi & ai \\ ai & -bi \end{pmatrix} = \begin{pmatrix} -bi & ai \\ ai & bi \end{pmatrix} \Rightarrow b = 0 \Rightarrow T = a\mathbb{1}_2$$

$$\Rightarrow T(\vec{z}) = \alpha \vec{z}, \quad \alpha \in \mathbb{C}$$

P 10. $D_3 \cong S_3$ $\qquad D = \langle a, b \mid a^2 = b^3 = (ab)^2 = 1 \rangle$

$\varphi: D_3 \to S_3$

$\varphi(a) = (12)$

$\varphi(b) = (123)$

P 11. (1) $\phi = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ n & n-1 & n-2 & \cdots & 1 \end{pmatrix}$

$$= \begin{cases} (1\ n)(2\ n-1)\cdots(\frac{n-1}{2}\ \frac{n+3}{2}) & n \text{ odd} \\ (1\ n)(2\ n-1)\cdots(\frac{n}{2}\ \frac{n}{2}+1) & n \text{ odd} \end{cases}$$

$$= \prod_{i=1}^{\lceil \frac{n-1}{2} \rceil} (i, n+1-i)$$

(2) $\lceil \frac{n-1}{2} \rceil$ even $\Leftrightarrow$ $n = 4k, 4k+1$ $(k \in \mathbb{N})$

odd $\Leftrightarrow$ $n = 4k+2, 4k+3$

(3) recall from lecture that

$$(i\,j) = (i, i+1)(i+1 . j)(i, i+1) \quad (i < j-1)$$

$$= \sigma_i (i+1, j) \sigma_i$$

$$= \sigma_i \sigma_{i+1} (i+2\ j) \sigma_{i+1} \sigma_i$$

alternatively.

$$\phi = (n-1 . n)(n-2 . \cdots n) \cdots (1\,2\,3\,4 - n)$$

and $(i, i+1, \cdots j+1) = \sigma_i \sigma_{i+1} \cdots \sigma_j$