P1. (1) uniquness of e

$e: \quad eg = ge = g \quad \forall g \in G$

$e_1 e_2 = e_2 e_1 = e_1 = e_2$

(2) uniquness of inverse $\quad g \cdot g^{-1} = g^{-1} \cdot g = e$

$ab = ba = ac = ca = e$

$b = b(ac) = (ba)c = c$

P2. $H_1 \subset G$. $H_2 \subset G$ subgroups

(1) $H_1 \cap H_2$?

① $\exists e$? $\quad e \in H_1,\ e \in H_2 \Rightarrow e \in H_1 \cap H_2$ ✓

② $\exists h^{-1}$? $\quad \left.\begin{array}{l} h \in H_1 \Rightarrow h^{-1} \in H_1 \\ h \in H_2 \Rightarrow h^{-1} \in H_2 \end{array}\right\} \Rightarrow h^{-1} \in H_1 \cap H_2$ ✓

③ closure? $\quad \forall h_1, h_2 \in H_1 \cap H_2 \Rightarrow h_1, h_2 \in H_1$ & $h_1, h_2 \in H_2$

$\Rightarrow h_1 \cdot h_2 \in H_1,\ h_1 \cdot h_2 \in H_2 \Rightarrow h_1 h_2 \in H_1 \cap H_2$ ✓

$\Rightarrow H_1 \cap H_2$ is a subgroup

(2) $H_1 \cup H_2$?

Suppose $\exists h_1, h_2 \in H_1 \cup H_2$ s.t. $\boxed{\begin{cases} h_1 \in H_1,\ h_1 \notin H_2 \\ h_2 \in H_2,\ h_2 \notin H_1 \end{cases}}$

If $h_3 = h_1 \cdot h_2 \in H_1 \cup H_2$. then

$h_3 \in H_1$ and/or $h_3 \in H_2$. WLOG. assume it's H_1

then $h_2 = h_1^{-1} \cdot h_3 \in H_1$, contradices with the assumption in $\boxed{}$. which means one of them should not hold.

$\Rightarrow H_1 \subset H_2$, or $H_2 \subset H_1$

P3. $\forall a, b, ab \in G$. $a = a^{-1}$. $b = b^{-1}$

$$\Rightarrow (ab)^2 = (ab)(ab) = (ab)(a^{-1}b^{-1}) = e$$

$$\Rightarrow ab = (a^{-1}b^{-1})^{-1} = ba$$

P4. $(H \text{ is a subgroup}) \overset{\text{iff}}{\Longleftrightarrow} (e \in H \ \& \ h_1, h_2 \in H \Rightarrow h_1 h_2^{-1} \in H)$

$\Rightarrow$ trivial by def. of (sub)group.

$\Leftarrow$ $e, h \in H$. $\Rightarrow$ $e \cdot h^{-1} = h^{-1} \in H$ (exists inverse)

$h_1 . h_2 \in H$. $\Rightarrow$ $h_2^{-1} \in H$. $\Rightarrow$ $h_1 (h_2^{-1})^{-1} = h_1 h_2 \in H$.

(closure)

P5 $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SU(2)$ g unitary: $\langle gx, gy \rangle = \langle x, y \rangle$

$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ \gamma \end{pmatrix}$ and $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \beta \\ \delta \end{pmatrix}$ orthonormal

$$|\alpha|^2 + |\gamma|^2 = |\beta|^2 + |\delta|^2 = 1$$

$\Rightarrow \bar{\alpha}\beta + \bar{\gamma}\delta = 0 \quad \Rightarrow \bar{\alpha} = \lambda\delta \quad \bar{\gamma} = -\lambda\beta \qquad \lambda \in \mathbb{C}$

$$\Rightarrow g = \begin{pmatrix} \bar{\lambda}\bar{\delta} & \beta \\ -\bar{\lambda}\bar{\beta} & \delta \end{pmatrix} \Rightarrow \det g = \bar{\lambda}(|\delta|^2 + |\beta|^2) = 1$$

$\Rightarrow \bar{\lambda} = 1$

$$\Rightarrow g = \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} \quad \text{with } |z|^2 + |w|^2 = 1 \qquad \begin{pmatrix} \delta = \bar{z} \\ \beta = -\bar{w} \end{pmatrix}$$

P6. Canonical transformations

(1) trivial.

(2) Def in lecture

$$Sp(2n, K) := \{ A \in GL(2n, K) \mid A^T J A = J \}$$

equiv. $AJA^T = J$

$$J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \qquad J = J^* = -J^T = -J^{-1}$$

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -A_{12} & A_{11} \\ -A_{22} & A_{21} \end{pmatrix} \begin{pmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \end{pmatrix}$$

$$= \begin{pmatrix} A_{11}A_{12}^T - A_{12}A_{11}^T & A_{11}A_{22}^T - A_{12}A_{21}^T \\ A_{21}A_{12}^T - A_{22}A_{11}^T & A_{21}A_{22}^T - A_{22}A_{21}^T \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\boxed{\begin{aligned}&\Rightarrow (A_{11}A_{12}^T - A_{12}A_{11}^T)_{ij} = 0 \quad \forall\, i j \in [1, n] \\ &(A_{11}A_{22}^T - A_{12}A_{21}^T)_{ij} = \delta_{ij}\end{aligned}}$$

$$\begin{pmatrix}\vec{Q}\\ \vec{P}\end{pmatrix} = \begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}\begin{pmatrix}\vec{q}\\ \vec{p}\end{pmatrix} = \begin{pmatrix}A_{11}\vec{q} + A_{12}\vec{p}\\ A_{21}\vec{q} + A_{22}\vec{p}\end{pmatrix}$$

$$Q_i = \sum_{i=1}^{n}(A_{11})_{ij}\, q_j + \sum (A_{12})_{ij}\, p_j$$

$$P_i = \sum_{i=1}^{n}(A_{21})_{ij}\, q_j + \sum (A_{22})_{ij}\, p_j$$

$$\frac{\partial Q_i}{\partial q_l} = (A_{11})_{il} \qquad \frac{\partial Q_i}{\partial p_l} = (A_{12})_{il}$$

$$\frac{\partial P_i}{\partial q_l} = (A_{21})_{il} \qquad \frac{\partial P_i}{\partial p_l} = (A_{22})_{il}$$

$$\{Q_i, Q_j\} = \sum_l \left(\frac{\partial Q_i}{\partial q_l}\frac{\partial Q_j}{\partial p_l} - \frac{\partial Q_i}{\partial p_l}\frac{\partial Q_j}{\partial q_l}\right) = \sum_l \big[(A_{11})_{il}(A_{12})_{jl} - (A_{12})_{il}(A_{11})_{jl}\big]$$

$$= \underline{(A_{11}A_{12}^T - A_{12}A_{11}^T)_{ij} = 0} \quad \checkmark$$

$\{P_i, P_j\}$ is similar.

$$\{Q_i, P_j\} = \sum_l \left(\frac{\partial Q_i}{\partial q_l}\frac{\partial P_j}{\partial p_l} - \frac{\partial Q_i}{\partial p_l}\frac{\partial P_j}{\partial q_l}\right)$$

$$= \sum_l \big[(A_{11})_{il}(A_{22})_{jl} - (A_{12})_{il}(A_{21})_{jl}\big]$$

$$= \underline{(A_{11}A_{22}^T - A_{12}A_{21}^T)_{ij} = \delta_{ij}} \quad \checkmark$$