Semester review

GT

A. Groups $\qquad$ B. Group rep.

A. Groups

1 Def. of groups (G, m, I, e)

① set G.

② $\underline{m}$: $G \times G \to G$

③ $\underline{I}$: $G \to G$

} closure

④ $e \in G$. $g \cdot e = e \cdot g = g$ $\quad \underline{m}(g, \underline{I}(g)) = e$

∘ $\underline{m}$. $\underline{I}$: $\mathbb{Z}$. $\mathbb{R}$. $\mathbb{C}$. $\quad$ "+" , "×" ×

∘ finite vs. infinite : $|G| \overset{?}{<} \infty$

∘ abelian vs non-abelian $\quad ab \overset{?}{=} ba$

semidirect & direct product

a. semi. $H \rtimes_\alpha G \qquad \alpha : \mathrm{Aut}(H)$

$$(h_1, g_1) \cdot_\alpha (h_2, g_2) := (h_1 \underline{\alpha_{g_1}(h_2)}, g_1 g_2)$$

6. α trivial $\quad (h_1, g_1)\cdot(h_2, g_2) = (h_1h_2, g_1g_2)$

<u>subgroups</u> $\quad H \subset \underline{G}$

$$\left.\begin{array}{ll} \underline{m}: & H\times H \to H \\ \underline{I}: & H \to H \end{array}\right\}$$

$\{e\}$, G. $\quad$ proper $H \neq G$.

↳ $H \triangleleft G$. : $gHg^{-1} = H$

↳ simple group.

↳ <u>centralizer</u>

$C_G(h) = \{g \in G.\ gh = hg\}$

$C_G(H) = \{g \in G,\ gh = hg.\ \forall h \in H\}$

↳ $C_G(G) =: \underline{Z(G)}$

<u>normalizer</u>:

$N_G(H) := \{g \in G.\ \underline{gHg^{-1} = H}\}$

$C_G(H) \subseteq N_G(H)$

<u>$GL(n,k)$</u>:

<u>subgroups</u>: 1 $\quad SL$

$$\left.\begin{array}{ll} SL & \\ O. & SO \\ U. & SU \end{array}\right\} \det M = ?$$

② $\underline{A^T J A = J}$

↪ $O(p,q)$ $\quad J_{pq} = \begin{pmatrix} -1_p & 0 \\ 0 & 1_q \end{pmatrix}$

→ $O(1,3)$

symplectic $\quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

<u>group presentation</u>.

$$G = \langle \underbrace{g_1 \cdots g_n}_{\text{generator}} \mid \underbrace{R_1 \cdots R_r}_{\text{relations}} \rangle$$

μ_N : $\langle A \mid A^N = 1 \rangle$

D_n : $\langle A, B \mid A^n = B^2 = (AB)^2 = 1 \rangle$

$\mathbb{Z}$: $\langle 1 \rangle$

2. Homomorphism & isomorphism :

$\varphi : G \to G'$

$$\begin{array}{ccc} G \times G & \xrightarrow{m} & G \\ {\scriptstyle \varphi\times\varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ G' \times G' & \xrightarrow{m'} & G' \end{array}$$

$\varphi(g_1) \cdot_{G'} \varphi(g_2) = \varphi(g_1 \cdot_G g_2)$

↪ $\begin{cases} \varphi(e) = e' \\ \varphi(g^{-1}) = \varphi(g)^{-1} \end{cases}$

$\ker\varphi = \{ g \in G : \varphi(g) = 1_{G'} \}$

$\operatorname{im}\varphi = \varphi(G)$

Ex. ① $\pi : SU(2) \to SO(3)$

$\ker\pi = \{\pm 1\}$

$\operatorname{im}\pi = SO(3)$

② $T : G \to GL(V)$

isomorphism : hom + (1-1 & onto invertible)

1-1: $\ker\varphi = \{e\}$

onto: $\varphi(G) = G'$

↪ $G = G'$: $Aut(G)$

Ex. ① $\mu_N \cong \mathbb{Z}_N$

② $GL(V) \cong GL(n, K)$

matrix rep: $T : G \to GL(n, K)$

$T(g)\hat{e}_i = T(g)_{ji}\hat{e}_j$

↪ $T \cong T'$ $\exists S \in GL(n, K)$

$T'(g) = S\,T(g)\,S^{-1}$ $(\forall g \in G)$

3. Group action G on X

$\Phi : G \to S_X := \{ X \xrightarrow{f} X \ , \ f \text{ invertible} \}$

$g \mapsto \phi(g, \cdot)$

$\Phi_g(x) = \phi(g, x) =: g \cdot x$

$g_1 \cdot (g_2 \cdot x) = (g_1 g_2) x$

↳ orbits $\quad Orb_G(x) = \{ g \cdot x \ , \ g \in G \}$

① $x \sim y$ iff $g \cdot x = y$

② partition of G

$O_G(x) = O_G(x')$ or $O_G(x) \cap O_G(x') = \phi$

set X/G

↳ fixed points $\quad Fix_X(g) = \{ \underbrace{gx = x}_{x \in X} \} \subset X$

↳ Stabilizer $\quad Stab_G(x) = \{ g \in G : g \cdot x = x \} \subset G$

group action is :

1. effective : $Fix_X(g \neq e) \neq X$
2. transitive : $Orb_G(x) = X$
3. free : $Fix_X(g \neq e) = \phi$

Theorem (Stab - orbit)

$$O_G(x) \xrightarrow{\cong} G/G^x$$

$$g\cdot x \longmapsto g\cdot G^x$$

finit G: $|O_G(x)| = [G : G^x]$

Ex. $SO(3)$ on S^2

$$S^2 \cong SO(3)/SO(2)$$

$$S^2 \cong SU(2)/U(1)$$

4. G action on G.

① $H \subset G$. right action on G

$$gH = \{gh : h \in H\}$$

$\hookrightarrow$ $\underline{g_1 H = g_2 H}$ $\quad g_1 H \cap g_2 H = \emptyset$

(Langrange) ~~Finite~~ G.

$$|G|/|H| = [G : H]$$

② action by conjugacy

$h \sim h'$ if $h' = ghg^{-1}$

$C(h) = \{ghg^{-1} : g \in G\} = h^G$

$$|C(g)| = [G : \underline{C_G(g)}]$$

$\hookrightarrow$ $Stab_G(g)$

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Finite G: $|C(g)| = \dfrac{|G|}{|C_G(g)|}$

$+ \ \sum' |C(g)| = |G|$

$\Rightarrow$ class eg.

$$|G| = \sum_{\{C_g\}} \frac{|G|}{|C_G(g)|}$$

$\hookrightarrow$ ① $|G| = p^n \rightarrow Z(G) \neq \{e\}$

② (Cauchy)

$p \mid |G| \Rightarrow \exists g.$ order p

$\hookrightarrow$ class function : f on G

$$f(g g_0 g^{-1}) = f(g_0) \quad \forall g, g_0 \in G$$

$\hookrightarrow$ $\chi_T(g) = \mathrm{Tr}\ T(g)$

5. morphisms of G-spaces / equvariant map

$f: X \to X'$

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \Phi(g) \downarrow & & \downarrow \Phi'(g) \\ X & \xrightarrow{f} & X' \end{array}$$

6. <u>S_n</u>

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 3 \end{pmatrix} =: (1\,2\,4\,3)$$

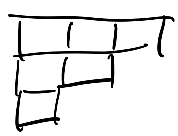
① $\phi \in S_n$: unique cycle decomposition

② r-cycles conjugate

↳ conj. class labeled as eg

$\vec{\lambda} = \{3, 2, 1\}$

Young diagram



S_6

$$\mathrm{sgn}: S_n \longrightarrow \mathbb{Z}_2$$
$$\phi \longmapsto \mathrm{sgn}(\phi) := (-1)^{n-t}$$

$A_n \lhd S_n$ $\quad (\longleftarrow H \subset G.\ [G:H] = 2)$

$|A_n| = \frac{1}{2} |S_n|$

7. <u>quotient groups</u>

$N \lhd G$: G/N

$$(g_1 \cdot N) \cdot (g_2 \cdot N) := (g_1 g_2) N$$

$$\mu : G \to G/N$$
$$g \longmapsto gN$$

$\ker \mu = N$

Theorem: $G/\ker\mu \cong \operatorname{im}\mu$

Ex. $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$ $\quad (\mu : i \mapsto i + n\mathbb{Z})$

SES: $1 \to \ker\mu \to G \xrightarrow{\mu} \operatorname{im}\mu \to 1$

ES: $\to G_{i-1} \xrightarrow{f_{i-1}} G_i \xrightarrow{f_i} G_{i+1}$

$\underline{\ker f_i = \operatorname{im} f_{i-1}}$

$1 \to N \to G \to Q \to 1$

- $N \cong H \triangleleft G$
- $Q \cong G/H$

$1 \to \mathbb{Z}_2 \to SU(2) \xrightarrow{\pi} SO(3) \to 1$

$A \subset Z(E)$ $\quad E \quad\quad G$

B. Group rep.

1. Def. ① $G \to GL(V) \cong GL(n, K)$

$g \mapsto T(g) \longmapsto U(g)$

② equivalent rep

$$\begin{tikzcd} V_1 \arrow[r, "A"] \arrow[d, "T_1(g)"'] & V_2 \arrow[d, "T_2(g)"] \\ V_1 \arrow[r] & V_2 \end{tikzcd}$$

$\Rightarrow T_2(g) = A T_1(g) A^{-1}$

③ unitary.

$\langle \underline{U(g)} w, \underline{U(g)} v \rangle = \langle w, v \rangle$

↪ compact / finite

2. Haar measure:

$f : G \to \mathbb{C}$

$\int_G dg : f \longmapsto \langle f \rangle$

$\int_G f(hg) dg = \int_G f(g) dg$ left-inv.

G. finite / compact Left = right

Ex. $\mathbb{R}$, $\mathbb{R}^*_{>0}$ $\int \frac{dx}{x}$

$SU(2)$ $\int \frac{1}{16\pi^2} d\varphi \, d\psi \, \sin\theta \, d\theta$

unitarization :

$$\langle v, w\rangle_2 := \int_G \langle T(g)v, T(g)w\rangle_1 \, dg$$

3. Regular rep $\quad f \in \mathrm{Map}(G, \mathbb{C})$

$$[(g_1, g_2) f](h) = f(g_1^{-1} h g_2)$$

$$((g_1, g_2) \cdot g_0 = g_1 g_0 g_2^{-1})$$

$$G \times G \longrightarrow \mathrm{End}(\{f\})$$

$$L^2(G) = \{ f : G \to \mathbb{C} \mid \underset{\langle f, f\rangle}{\underbrace{\int_G |f(g)|^2 dg}} < \infty \}$$

(Hilbert space)

$\hookrightarrow G \times \{1\}$ or $\{1\} \times G$

$$\hookrightarrow \quad \delta_g(g') = \begin{cases} 1 & g' = g \\ 0 & \text{otherwise} \end{cases}$$

$$g_1 \cdot \delta_{g_2} = \delta_{g_1 g_2}$$

4. reducible & irreducible.

$\exists W \subset V$ invariant subspace

completely : $V \cong W \oplus W^{\perp}$

Ex. ① Abelian.

② S_n rep on $\{ \vec{e}_i,\ i=1,\dots,n \} =: V$

$$V = W \oplus W^{\perp}$$
$$\ni$$
$$\{ \sum \hat{e}_i \}$$

③ $L^2(G)$

④ non-compact $\underline{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}}$ ×

↪ isotypic decomposition

$$V \cong \oplus_\mu \underline{\mathrm{Hom}_G(V^\mu, V)} \otimes V^\mu$$
$$\cong \oplus_\mu K^{a_\mu} \otimes V^\mu$$
$$=: \oplus_\mu a_\mu \underline{V}^\mu \qquad a_\mu = \dim_K \mathrm{Hom}_G(V^\mu, V)$$

5. Schur's lemma

$$\begin{array}{ccc} V_1 & \xrightarrow{A} & V_2 \\ T_1(g) \downarrow & & \downarrow T_2(g) \\ V_1 & \xrightarrow{A} & V_2 \end{array} \qquad \underline{V_1, V_2 \text{ . irreps}}$$

① $A = 0$. or an isomorphism

② $V_1 = V_2 = V$ (complex vec. space)

$$\underline{A(v) = \lambda v} \quad (\lambda \in \mathbb{C})$$

For physics: $\mathcal{H}$

$$[H, T(G)] = 0$$

$$\mathcal{H} \cong \bigoplus_{\mu} \underbrace{\mathrm{Hom}_G(\mathcal{H}^{\mu}, \mathcal{H})}_{\circlearrowleft\, D^{\mu}} \otimes \mathcal{H}^{\mu}$$

$$H \cong \underline{H^{\mu}} \otimes \mathbb{1}_{\mathcal{H}^{\mu}}$$

$$\tilde{H} = SHS^{-1} = \begin{pmatrix} \overset{e_1}{\square} & \overset{0}{} & 0 & & \\ & \overset{e_2}{\square} & 0 & & \\ & & \square & & \\ * & & & \square & \\ & & & & \square \end{pmatrix}$$

6. Pontryagin dual. S abelian

$$\hat{S} = \mathrm{Hom}(S, U(1)) \ni \chi_1, \chi_2$$

$$(\chi_1 \cdot \chi_2)(s) := \chi_1(s)\,\chi_2(s)$$

$$\hat{\hat{S}} = \mathrm{Hom}(\hat{S}, U(1))$$

(P.v.K.) $S \in LCA$: $\hat{\hat{S}} \cong S$

S	$\hat{S}$
$\mathbb{Z}_n$	$\mathbb{Z}_n$
$\mathbb{R}$	$\mathbb{R}$
$U(1)$	$\mathbb{Z}$
$\mathbb{Z}$	$U(1)$

$\hookrightarrow \quad \Gamma \cong \mathbb{Z}^d$

$$\hat{\Gamma} \cong U(1)^d \longrightarrow BZ.$$

$\underline{L_\gamma} \varphi(x) = \varphi(x+\gamma) \qquad \bar{k} \sim k + 2n\pi$

$\hookrightarrow = \chi_k(\gamma)\varphi(x) \qquad \gamma \in \Gamma$

$\varphi(x) = e^{2\pi i k x} u_k(x)$

$u_k(x) = u_k(x+\gamma)$

7. Peter – Weyl theorem & orthogonal relations

$$L^2(G) \cong \oplus_\mu \mathrm{End}(V^\mu)$$

finite G : $|G| = \sum_\mu n_\mu^2$

① $\langle \underline{M^{\mu_1}_{i_1, j_1}}, M^{\mu_2}_{i_2, j_2} \rangle = \frac{1}{n_\mu} \delta^{\mu_1 \mu_2} \delta_{i_1 i_2} \delta_{j_1 j_2}$

② $\iota : \oplus_\mu \mathrm{End}(V^\mu) \longrightarrow L^2(G)$

$\oplus_i s_i \longmapsto \underline{\Sigma_i \varphi_{s_i}}$

$\hookrightarrow \{\chi_\mu\}$ ON basis of $L^2(G)^{\text{class}}$

$$\int_G dg\, \overline{\chi_\mu(g)}\, \chi_\nu(g) = \delta_{\mu\nu}$$

$\hookrightarrow$ character table. row & column

↪ projectors $P^{\mu}_{ij} = n_\mu \int_G \overline{u^{\mu}_{ij}(g)}\, T(g)\, dg$

$$P^{\mu} = n_\mu \int_G \overline{\chi_\mu(g)}\, T(g)\, dg$$

$$\underline{P^{\mu} P^{\nu} = \delta_{\mu\nu} P^{\nu}}$$

↪ ① general finite group.

"class operator":

$$\underline{\hat{C}_i = \sum_{\mu=1}^{r} \lambda_i^{\mu} P^{\mu}}$$

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② S_n

$C = PQ$ ~~[illegible]~~

↪ Schur-Weyl duality

$$\underline{V}^{\otimes n} \cong \oplus D^{\lambda} \otimes V^{\lambda}$$

reps G irreps of S_n

V irrep. $\to$ D^{λ} irrep

$V = \mathbb{C}^d \to$ irrep $GL(d, \mathbb{C})$

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8. induced rep. $\Psi \in \mathrm{Map}(G, V)$

$$(g, h)\,\Psi(g_0) = \rho(h)\,\Psi(g^{-1} g_0 h)$$

$\underline{\mathrm{Ind}\,V}$ = fixed points $\{1\} \times H$

$$= \{ \Psi : G \to V \mid \Psi(g h^{-1}) = \rho(h)\,\Psi(g) \}$$

$$\begin{tikzcd} \underline{V_c} \arrow[r, "\rho_{\mathrm{Ind}}(g)"] \arrow[d, "ev_c"'] & V_{gC =: C'} \arrow[d, "ev_{c'}"] \\ \underline{V} \arrow[r, "\rho_V(g_{c'}^{-1} g g_c)"'] & V \end{tikzcd}$$

$C = C'$

$$\chi_{\mathrm{Ind}} = \sum_{gC = C} \chi_V(g_c^{-1} g g_c)$$

$\mathrm{Ind}\,V \cong \bigoplus \underline{V_c}$ $\quad$ $\dim = [G : H] \dim V$