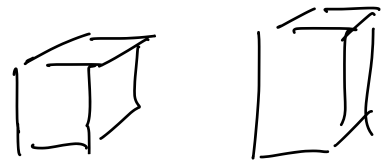
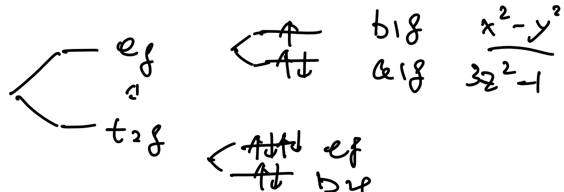
Application

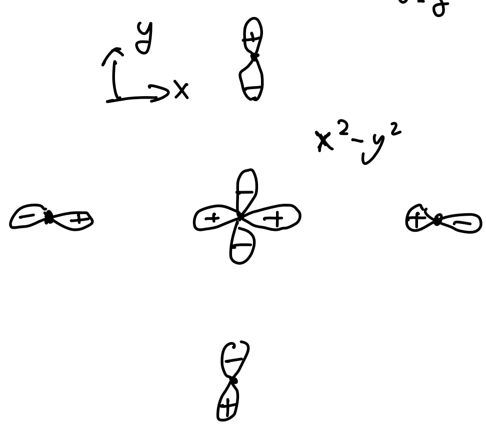
O(3) Oh D_{4h} Cu^{2+} : $\underline{d^9}$

l —

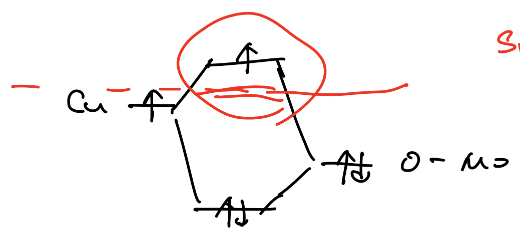


$l=2$





Zhang-Rice singlet



$$H = -t \sum c_{i\sigma}^{+} c_{j\sigma} + U \sum n_{i\uparrow} n_{i\downarrow}$$

Single-band Hubbard model

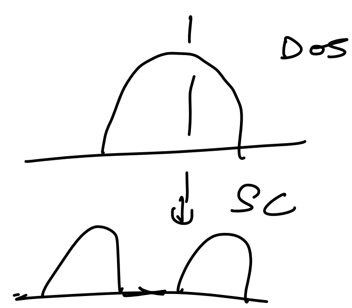
$$H = \sum_{k\sigma} \epsilon_k c_{k\sigma}^{+} c_{k\sigma} - V \sum_{kk'} \underline{c_{k\uparrow}^{+} \; c_{-k\downarrow}^{+}} \; \underline{c_{-k'\downarrow} \; c_{k'\uparrow}}$$

$\langle c^{+} c \rangle$

SC $\quad \Delta_k = \underline{c_{-k\downarrow} c_{k\uparrow}} \quad \langle \quad \rangle$

$$H_k = \begin{pmatrix} \epsilon_k & -\Delta_k \\ -\bar{\Delta}_k & -\epsilon_k \end{pmatrix}$$

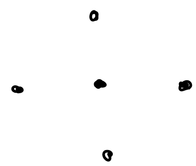
$$\underline{\xi(k)} = \pm \sqrt{\underline{\epsilon_k^2} + \underline{\Delta_k^2}}$$



Δ_k ?

$\Gamma = (0, \pm 1)$

$(\pm 1, 0)$



$$\Delta_k = \sum_r \underline{\underline{\Delta_r}}\, e^{i\vec{k}\vec{r}}$$

$\leftarrow$ scaler

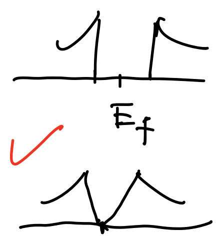
Δ_0 . $\underline{\{e^{\pm i k_x}, e^{\pm i k_y}\}}$ $\{e^{i(k_x \pm k_y)}, e^{-i(k_x \pm k_y)}\}$

① $\underline{P^\mu} = n_\mu \int_G \overline{\chi^\mu(g)}\, T(g)\, dg$ ✓

② $T\underline{(g)}$

③ $\underline{P^\mu} \Delta_k = \underline{\Delta_k}$

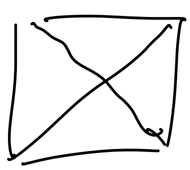
STM



A_1 : $\cos k_x + \cos k_y$ "s"

B_1 : $\cos k_x - \cos k_y$ "d" ✓

E : $[\sin k_x, \sin k_y]$ "p-wave"



$\Delta = 0$ if $\cos k_x = \cos k_y$

Review of representation theory

1. Defs. ① $G \to GL(V) \cong GL(n, K)$

$$g \longmapsto T(g) \longmapsto M(g)$$

$$T(g)\hat{e}_i = \sum_i M(g)_{ji}\hat{e}_j$$

② equivalent rep.

$$\begin{array}{ccc} V_1 & \xrightarrow{A} & V_2 \\ T_1(g)\downarrow & & \downarrow T_2(g) \\ V_1 & \xrightarrow{A} & V_2 \end{array}$$

invertible intertwiner A

$$T_2(g) = A\,T_1(g)\,A^{-1}$$

is an iso morphism

③ unitary rep: V is an inner product space

$$\langle U(g)w, U(g)v\rangle = \langle w, v\rangle$$

$$(\forall w, v \in V)$$

2. Haar measure:

$$f: G \to \mathbb{C} \in Map(G, \mathbb{C})$$

$$\int_G f(hg)\,dg = \int_G f(g)\,dg \qquad (\forall h \in G)$$

G : finite / compact

left haar measure = right.

Ex. $G = \mathbb{R}$ $\quad \int dx$

$G = \mathbb{R}^*_{>0}$ $\quad \int \frac{dx}{x}$

$G = GL(n, \mathbb{R})$ $\quad \int \underline{|\det g|}^{-n} \prod_{ij} dg_{ij}$

$G = SU(2)$ $\quad \int \frac{1}{16\pi^2} d\varphi\, d\phi \sin\theta\, d\theta$

$[0, 4\pi)$ $\quad [0, 2\pi)$ $\quad [0, \pi)$

Unitarization $\quad \langle v, w \rangle_2 = \int_G \langle T(g)v, T(g)w \rangle_1 \, dg$

3. Regular representation: $G \times G$ action on G.

$$(g_1, g_2) \longmapsto L(g_1) R(g_2^{-1})$$

$$(g_1, g_2) \cdot g_0 = g_1 g_0 g_2^{-1}$$

$f \in Map(G, \mathbb{C})$.

$$[(g_1, g_2) \underline{f}](h) = f(g_1^{-1} \underline{h} g_2)$$

$\{f\}$ is a rep of $\underline{G \times G}$.

Regular rep

$$L^2(G) := \{ f : G \to \mathbb{C} \mid \int_G |f(g)|^2 dg < \infty \}$$

$\langle f, f \rangle < \infty$

(Hilbert space)

$G \times \{1\}$, $\{1\} \times G$

finite group. "δ-basis".

$$\delta_g(g') = \begin{cases} 1 & g' = g \\ 0 & \text{otherwise} \end{cases}$$

$$(g_1 \cdot \delta_{g_2})(g) = \delta_{g_2}(g_1^{-1} g) = \delta_{g_1 g_2}(g)$$

$$\underline{g_1 \cdot \delta_{g_2} = \delta_{g_1 g_2}}$$

4. reducible & irreducible reps.

if $\exists W \subset V$ a proper, nontrivial

<u>invariant subspace</u> $(W \neq 0, V)$

$(\forall w \in W, \forall g \quad T(g)w \in W)$

completely reducible. $V \cong \oplus W^{\mu}$

Ex. ① Abelian groups

② canonical rep of S_n $\{\vec{e}_i\}$ $(\mathbb{R}^n)$

$$\sigma \cdot \hat{e}_i = \hat{e}_{\sigma(i)}$$

$$W = \Sigma \hat{e}_i$$

$$V = W \oplus W^\perp$$

isotypic decomposition

$$V \cong \oplus_\mu \underline{a_\mu} V^\mu$$

5. Schur's lemma:

V_1, V_2 irrep

$$\begin{tikzcd} V_1 \arrow[r, "A"] \arrow[d, "T_1(g)"'] & V_2 \arrow[d, "T_2(g)"] \\ V_1 \arrow[r, "A"] & V_2 \end{tikzcd}$$

① A is 0 or isomorphism

② $V_1 \cong V_2 = V$. a complex vector space

$$A(v) = \lambda v \qquad (\lambda \in \mathbb{C})$$

$SO(2)$

$\hookrightarrow [H, T(G)] = 0$

6. Pontryagin dual. Abelian S

$$\hat{S} := \mathrm{Hom}(S, U(1))$$

$$(\chi_1 \cdot \chi_2)(s) = \chi_1(s) \cdot \chi_2(s) \qquad s \in S$$

LCA. $\hat{\hat{S}} \cong S$

S	$\hat{S}$	$\hat{\hat{S}}$
$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$
$U(1)$	$\mathbb{Z}$	$U(1)$
$\mathbb{Z}_n$	$\mathbb{Z}_n$	$\mathbb{Z}_n$
$\mathbb{Z}$	$U(1)$	$\mathbb{Z}$

$\hookrightarrow$ Bloch's theorem

$$\underline{L_\gamma}\, \varphi(x) = \varphi(x+\gamma) = \underline{\chi_{\vec{k}}(\gamma)}\, \varphi(x)$$

$$\begin{cases} \varphi(x) = e^{2\pi i k \cdot x} u_k(x) \\ u_k(x) = u_k(x+\gamma) \end{cases}$$

7. Peter-Weyl theorem : orthogonal relations

between matrix elements

characte

f

$L^2(G) \simeq \oplus_\mu \mathrm{End}(V^\mu)$

n-dim $\to$ $\left(\quad \right)^{n\times n}$

$\dim L^2(G) = |G| = \sum n^2$

V^μ. $\dim V^\mu = n_\mu$

$\{M^{\mu_i}_{jk}\}$ complete basis of $L^2(G)$

$$\boxed{\langle M^{\mu_1}_{i_1 j_1}, M^{\mu_2}_{i_2 j_2} \rangle = \frac{1}{n_\mu} \delta^{\mu_1 \mu_2} \delta_{i_1 i_2} \delta_{j_1 j_2}}$$

$$\boxed{|G| = \sum_\mu n_\mu^2}$$

finite G.

$\hookrightarrow$ $\{\chi_\mu\}$ ON basis of $L^2(G)^{\text{class}}$

$$\boxed{\begin{aligned} &\frac{1}{|G|} \sum_{\{C_i\}} m_i \overline{\chi_\mu(C_i)}\, \chi_\nu(C_i) = \delta_{\mu\nu} \\ &\frac{m_i}{|G|} \sum_\mu \overline{\chi_\mu(C_i)}\, \chi_\mu(C_j) = \delta_{ij} \end{aligned}}$$

$$P^\mu_{ij} = n_\mu \int_G \overline{M^\mu_{ij}(g)}\, T(g)\, dg$$

$\underline{c^2 = \lambda c}$

S_n : $c = PQ$

$P = \sum_{p \in R(T)} p$

$Q = \sum_{g \in C(T)} \mathrm{sgn}(g)\, g$

Schur – Weyl duality

$$V^{\otimes n} \cong \bigoplus D^{\lambda} \otimes V^{\lambda}$$

irrep. $GL(n)$ ← irreps of S_n