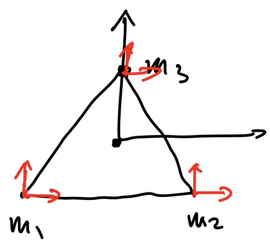
HW. P. 28 S_3 irreps V^+. V^-. V^2

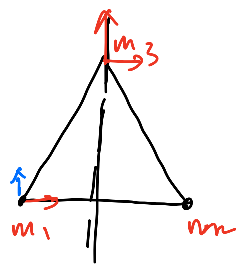
① $\chi_{V_1 \otimes V_2} = \chi_{V_1} \chi_{V_2}$

② $n_{V_i} = \langle \chi_{V_i} . \chi_V \rangle$

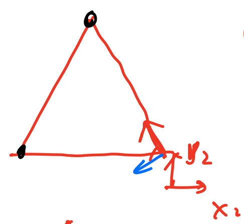
P29,



$\varphi = (\delta x_1, \delta y_1, \delta x_2, \delta y_2, \delta x_3, \delta y_3)$



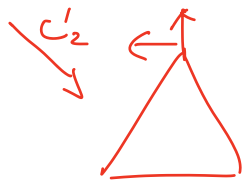
C_3



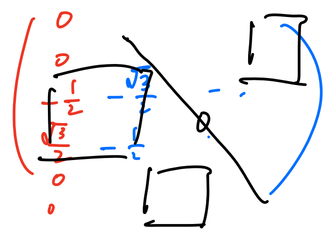
$(1,0,0,0,0,0)^T$

$\rightarrow (0,0,-\frac{1}{2},\frac{\sqrt{3}}{2},0,0)^T$

C_2'



$M(C_3) =$



$\chi_V(E) = 6$

$\chi_V(C_3) = 0$

$\chi_V(C_2') = 0$

$\delta x_3 \rightarrow -\delta x_3$

$\delta y_3 \rightarrow \delta y_3$

$n_p = \langle \chi_p . \chi_V \rangle$

$D_3 \cong S_3$	E	$2C_3$	$3C_2'$
A_1	1	1	1
A_2	1	1	-1
E	2	-1	0
V	6	0	0

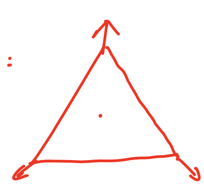
$n_{A_1} = \frac{1}{6} \times 6 \times 1 = 1$

$n_{A_2} = \frac{1}{6} \times 6 \times 1 = 1$

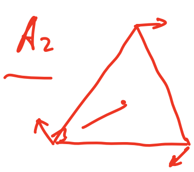
$n_E = \frac{1}{6} \times 6 \times 2 = 2$

$\boxed{V \cong A_1 \oplus A_2 \oplus 2E}$

A_1:



A_2



P30. $D_4 = \langle r, s \mid r^4 = s^2 = (rs)^2 = 1 \rangle$

$$= \{1, r, r^2, r^3, s, \underset{sr^3}{\underset{\shortparallel}{rs}}, \underset{sr^2}{\underset{\shortparallel}{r^2s}}, \underset{sr}{\underset{\shortparallel}{r^3s}}\}$$

Rep. $U(n)$ $\quad SU(2)$ irreps: $\overbrace{\square\square\square\square\square}^{l}$

$$\begin{array}{c} 1 \\ 2 \end{array} \qquad \left.\begin{array}{c} 1\,1\,1\,1\,1 \\ 1\,1\,1\,1\,2 \\ \vdots \\ 2\,2\,2\,2\,2 \end{array}\right\} l+1$$

$l = 2j$ "spin-j representation".

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9. Lie algebra and Lie groups;

irreps of $SO(3)$ and $SU(2)$

References.

✓ ① Ramond, Group theory. A physicist's survey

Chap 5. onwards

✓ ② Das & Okubo. Lie groups and Lie algebras for physicists

Chap. 3–5

③ Fulton & Harris. Representation theory (GTM129)

Part II.

Lie group. group and a differentiable manifold.

$GL(n, K)$ and subgroups $SO(N)$, $SU(N)$.

Let us consider $SO(3)$. rotation in 3D.

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = e^{\theta J_3}$$

$$J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Similary. $R_x(\theta) = e^{\theta J_1}$ $\quad R_y(\theta) = e^{\theta J_2}$

$$\underline{J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}} \qquad J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

Arbitrary rotation is given by $\underline{e^{\theta \hat{n}\cdot\vec{J}}}$

$SO(3)$ is non abelian.

consider infinitesmal rotations around identity.

$$e^{\epsilon_2 J_2} e^{\epsilon_1 J_1} \simeq (1 + \epsilon_2 J_2 + \tfrac{1}{2}\epsilon_2^2 J_2^2)(1 + \epsilon_1 J_1 + \tfrac{1}{2}\epsilon_1^2 J_1^2)$$

$$= 1 + \epsilon_1 J_1 + \epsilon_2 J_2 + \underline{\epsilon_1\epsilon_2 J_2 J_1} + \tfrac{1}{2}\epsilon_1^2 J_1^2 + \tfrac{1}{2}\epsilon_2^2 J_2^2 + O(\epsilon^3)$$

$$e^{\epsilon_1 J_1} e^{\epsilon_2 J_2} \simeq 1 + \epsilon_1 J_1 + \epsilon_2 J_2 + \underline{\epsilon_1 \epsilon_2 J_1 J_2} + \cdots$$

The difference $= \epsilon_1\epsilon_2 (J_1 J_2 - J_2 J_1)$

$$[J_1, J_2] := J_1 J_2 - J_2 J_1 = J_3$$

$$[J_i, J_j] = \epsilon_{ijk} J_k$$

J_1, J_2, J_3 are generators of some algebra.

"Lie algebra" $\Longleftrightarrow$ Lie group.

$$J_1, J_2, J_3 \quad \underset{\text{diff.}}{\overset{\exp}{\rightleftarrows}} \quad e^{\hat{n}\cdot\vec{J}}$$

$$\frac{d}{dt}(e^{tJ_i})\Big|_{t=0} = J_i \cdot e^{tJ_i}\Big|_{t=0} = J_i$$

reps of algebra $\Longleftrightarrow$ reps of group.

Definition A Lie algebra L is a vector space over a field K $(= \mathbb{R}, \mathbb{C})$. equiped with a bilinear map: $[\ ,\ ] : L \times L \to L$

that satisfies

① $\forall\, x, y \in L$

$$[x, y] = -[y, x] \in L$$

antisymmetric condition.

② $\forall\, x, y, z \in L$.

$$[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$$

Jacobi identity

①' The antisymmetric condition is equivalently expressed as

$[x, x] = 0$

$$0 = [x+y, x+y] = \cancel{[x,x]} + \underline{[x,y]} + \underline{[y,x]} + \cancel{[y,y]}$$

$$\Rightarrow [x,y] = -[y,x]$$

②' $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$

<u>Examples</u>.

1. $\mathfrak{gl}(N)$ (lower cases for algebra / upper group)

$M(N, K) \in \mathfrak{gl}(N)$ any $N \times N$ matrices

2. $\mathfrak{sl}(N)$

recall. $\underset{\sim}{SL}(N) := \{A \in GL(N, K) \mid \det A = 1\}$

$$\det(e^A) = \det(P^{-1} e^{u} P) = e^{\mathrm{tr} A} = 1$$

upper triangular / diagonal

$\mathrm{tr} A = 0$

$\mathfrak{sl}(N)$. $N \times N$ traceless matrices

3. $\mathfrak{so}(N)$

Consider infinitesmal rotation around $\mathbb{1}$.

$R = \mathbb{1} + \epsilon X \quad (e^{\epsilon X})$

$$\mathbb{1} = RR^T = (\mathbb{1}+\epsilon X)(\mathbb{1}+\epsilon X^T) + O(\epsilon^2)$$

$$= \mathbb{1} + \epsilon(X + X^T) + O(\epsilon^2)$$

$$\Rightarrow \quad X = -X^T$$

$(\underline{X_{ij} = -X_{ji}})$ skew-symmetric

$$[X, Y]_{ij} = X_{ik} Y_{kj} - Y_{il} X_{lj}$$

$$= (XY)_{ij} - Y_{li} X_{jl}$$

$$= (XY)_{ij} - (XY)_{ji}$$

$$[X, Y]_{ji} = -[X, Y]_{ij}$$

dimension of the vector space ?

$$d = \frac{1}{2} N(N-1)$$

$so(2)$ $\quad d = 1 \quad$ $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$e^{\theta J} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in SO(2)$$

$so(3)$ $\quad d = 3 \quad$ $\underline{J_1, J_2, J_3}$ as above

In QM. $\quad \vec{L} = \vec{r} \times \vec{p} \qquad L_{ij} = x_i p_j - x_j p_i =: \hat{L}_k$

$$\hat{p}_i = -i \frac{\partial}{\partial x_i}$$

$$[\hat{L}_1, \hat{L}_2] = \hat{L}_1 \hat{L}_2 - \hat{L}_2 \hat{L}_1$$

$$= (x_2 p_3 - x_3 p_2)(x_3 p_1 - x_1 p_3) -$$

$$(x_3 p_1 - x_1 p_3)(x_2 p_3 - x_3 p_2)$$

$$= -i x_2 p_1 + i x_1 p_2 = i \hat{L}_3$$

$$[\hat{L}_i, \hat{L}_j] = i \epsilon_{ijk} \hat{L}_k \qquad [-iJ_1, -iJ_2] = i(-iJ_3)$$

representations of $\tilde{L}_i$

① 1 dim vectorspace of C

$$\tilde{L}_i \cdot C = 0 \qquad (e^{\alpha \tilde{L}} C = (1 + \alpha \tilde{L} + \tfrac{1}{2}(\ )^2 + \cdots) C = C \ )$$

trivial representation "s orbital" $l = 0$

② $l=1$ vectorspace

$$\tilde{L}_1 \underline{x_i} = (x_2 p_3 - p_3 x_2) x_i$$
$$= -i(\underline{x_2} \delta_{i3} - \underline{x_3} \delta_{i2}) \in \text{span}\{x_i\}$$

$$\tilde{L}_1 = -i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\tilde{L}_2 = -i \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\tilde{L}_3 = -i \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} x_1 = r \sin\theta \cos\phi \\ x_2 = r \sin\theta \sin\phi \\ x_3 = r \cos\theta \end{cases} \qquad x_\pm = x_1 \pm i x_2 = r \sin\theta \, e^{\pm i\phi}$$

$$Y_1^{\pm}(\theta, \phi) = \mp \left(\frac{3}{8\pi}\right)^{\frac{1}{2}} \sin\theta \, e^{\pm i\phi} \quad \propto x_\pm$$

$$Y_1^{0}(\theta, \phi) = \left(\frac{3}{4\pi}\right)^{\frac{1}{2}} \cos\theta \quad \propto x_3$$

orthonormal: $\int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi \, Y_l^m \overline{Y_{l'}^{m'}} = \delta_{ll'} \delta_{mm'}$

③ $T_{ij} = x_i x_j = T_{ji}$ symmetric tensors / 2nd-order polynomial

$$\underline{L_{ij}}\,\underline{T_{kl}} = -i(\delta_{jk}\underline{T_{il}} + \delta_{jl}\underline{T_{ik}} - \delta_{ik}\underline{T_{jl}} - \delta_{il}\underline{T_{jk}})$$

span $\{T_{ij}\}$ is a rep of $\{L_{ij}\}$

$$\dim = \frac{1}{2}N(N+1)$$

This is a reducible rep.

$$L_{ij}T_{kk} = -2i(\delta_{jk}T_{ik} - \delta_{ik}T_{jk})$$

$$L_{ij}\sum T_{kk} = -2i(T_{ij} - T_{ji}) = 0$$

$\Rightarrow \sum T_{kk}$ is a trivial 1D rep.

The remaining $\frac{1}{2}N(N+1)-1$ is an irrep.

so(3): $\dim = \frac{1}{2}3\times 4 - 1 = 5 = \underline{2l+1}$ $l=2$

$$Y_{2,\pm2}(\theta,\phi) \propto \sin^2\theta\, e^{\pm 2i\phi} = x_\pm x_\pm$$

$$Y_{2,\pm1}(\theta,\phi) \propto \sin\theta\cos\theta\, e^{\pm i\phi} = x_\pm x_3$$

$$Y_{2,0}(\theta,\phi) \propto 3\cos^2\theta - 1 = 2x_3^2 - x_+x_-$$

$\Rightarrow$ $\boxed{\text{irreps of so(3) / SO(3) are of dimensions } 2l+1 \quad (l \in \mathbb{N})}$

4. $su(N)$

$$RR^{+} = (\mathbb{1}+\epsilon X)(\mathbb{1}+\epsilon X^{+}) = \mathbb{1}$$

$X = -X^{+}$ skew - Hermitian matrices

$X_{ij} = -X_{ji}^{*}$

$$[X, Y] = XY - YX = XY - Y^{+}X^{+} = XY - (XY)^{+}$$

$$\Rightarrow [X, Y] = -[X, Y]^{+}$$

For $N=2$. $su(2)$ consider 2-dim Hilbert space $\{|1\rangle, |2\rangle\}$
$\{|\uparrow\rangle, |\downarrow\rangle\}$

$$L_1 = \frac{1}{2}(|1\rangle\langle 2| + |2\rangle\langle 1|)$$

$$L_2 = \frac{i}{2}(|2\rangle\langle 1| - |1\rangle\langle 2|)$$

$$L_3 = \frac{1}{2}(|1\rangle\langle 1| - |2\rangle\langle 2|)$$

$$L_{\pm} = L_1 \pm i L_2 = \begin{cases} |1\rangle\langle 2| \\ |2\rangle\langle 1| \end{cases}$$

($|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$L_{+} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $L_{-} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $L_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$)

$$\begin{cases} [L_i, L_j] = i\epsilon_{ijk}L_k \\ [L_+, L_-] = 2L_3 \\ [L_3, L_\pm] = \pm L_\pm \end{cases}$$

$\{L_i\}$ do not commute

$L^2 = L_1^2 + L_2^2 + L_3^2$ commute with L_i $[L^2, L_i] = 0$

"total angular momentum"

"Caismir operator" in Lie algebra

choose the eigenstates of L^2, L_3 to be the basis of the representation.

$$\begin{cases} L_+ |j,m\rangle = \sqrt{(j-m)(j+m+1)}\, |j,m+1\rangle \\ L_- |j,m\rangle = \sqrt{(j+m)(j-m+1)}\, |j,m-1\rangle \\ L_3 |j,m\rangle = m |j,m\rangle \end{cases}$$

$$L^2 |j,m\rangle = j(j+1) |j,m\rangle \qquad m \in [-j, j]$$

$\dim \{ |j,m\rangle \} = 2j+1$.

$j = \frac{1}{2}$ $\qquad L_i \propto \frac{\sigma_i}{2}$

$j = 1$ $\qquad$ spin-1 rep

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}$$

$$L_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$$

$$L_3 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$SO(3) \cong SU(2)/Z_2$

$so(3) \simeq su(2)$

$SO(3) \neq SU(2)$

$SO(3) \sim \mathbb{R}P_3$

$SU(2) \sim S_3$