Recap.

irreps of S_n:

$$T = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \rightarrow \quad C(T) = P(T)\, Q(T)$$

$$P(T) = \sum_{p \in R(T)} p$$

$$Q(T) = \sum_{q \in C(T)} \mathrm{sgn}(q)\, q$$

$\underline{V} = K^d$ is the defining representation $\underline{GL(d, K)}$ (and $U(d)$)

$\underline{V^{\otimes n}}$ is a rep of S_n:

$$\sigma: \; v_i \otimes v_j \longrightarrow v_{\sigma(i)} \otimes v_{\sigma(j)}$$

Schur-Weyl duality:

$$\underline{\underline{V}}^{\otimes n} \cong \bigoplus_\lambda \underline{\underline{D_\lambda}} \otimes \underline{\underline{R_\lambda}}$$

λ labels a partition of n. / Young diagram.

$\Rightarrow$ The representations D_λ are irreducible representations of $GL(d, K)$ (and $U(d)$)

All irreps of $GL(d, K)$ can be obtained by varying $\underline{n}$.

$\Rightarrow$ Tensors of definite symmetries (obtained via Young symmetrizers) transform as irreps of $GL(d, \mathbb{K})$.

Example. $V^{\otimes 3} \overset{\text{span}}{=} \{ v_i \otimes v_j \otimes v_k \}$

S_3	$[()]$	$3[(12)]$	$2[(123)]$
1^+	1	1	1
1^-	1	-1	1
2	2	0	-1

$\chi([()]) = d^3$

$\chi([(12)]) = d^2$ $\qquad \underline{v_i \otimes v_i \otimes v_k}$

$\chi([(123)]) = d$

$$a_{1^+} = \langle \chi_{1^+}, \chi \rangle = \frac{1}{6}(d^3 \times 1 + d^2 \times 3 + d \times 2) = \underline{\frac{1}{6} d(d+1)(d+2)}$$

$$a_{1^-} = \langle \chi_{1^-}, \chi \rangle = \frac{1}{6}(d^3 - 3d^2 + 2d) = \frac{1}{6} d(d-1)(d-2)$$

$$a_2 = \langle \chi_2, \chi \rangle = \frac{1}{6}(2d^3 - 2d) = \frac{1}{3} d(d+1)(d-1)$$

① $\boxed{1 \,|\, 2 \,|\, 3}$ $\qquad C = PQ = e + (12) + (13) + (23) + (123) + (132)$

$$C \cdot V^{\otimes 3} = \operatorname{span} \{ \sum_\sigma v_{\sigma(i)} \otimes v_{\sigma(j)} \otimes v_{\sigma(k)} \}$$

$$= \operatorname{Sym}^3 V$$

[

$$t = \sum a_{ijk}\, v_i \otimes v_j \otimes v_k$$

$$\sigma \cdot t = \sum a_{ijk}\, v_{\sigma(i)} \otimes v_{\sigma(j)} \otimes v_{\sigma(k)}$$

$$= \sum a_{\sigma^{-1}(i)\sigma^{-1}(j)\sigma^{-1}(k)}\, v_i \otimes v_j \otimes v_k$$

$$\Rightarrow (\sigma \cdot a)_{ijk} = a_{\sigma^{-1}(i)\sigma^{-1}(j)\sigma^{-1}(k)}$$

]

$$\underline{(a_S)_{ijk} = \sum_\sigma a_{\sigma^{-1}(i),\sigma^{-1}(j),\sigma^{-1}(k)} = \sum_\sigma a_{\sigma(i)\sigma(j)\sigma(k)}}$$

$$\Rightarrow (a_s)_{jik} = (a_s)_{ijk}$$

$$(\sigma a_s)_{ijk} = (a_s)_{ijk}$$

② $\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}$ $\quad c = e - (12) - (13) - (23) + (123) + (132)$

$$(a_\wedge)_{ijk} = \sum_\sigma \mathrm{sgn}(\sigma)\, a_{\sigma^{-1}(i),\sigma^{-1}(j),\sigma^{-1}(k)}$$

$$\begin{aligned}
\underline{(a_\wedge)_{jik}} &= (\tau(ij)\, a_\wedge)_{ijk} \\
&= \sum_\sigma \tau(ij)\, \mathrm{sgn}(\sigma)\, a_{\sigma^{-1}(i)\,\sigma^{-1}(j)\,\sigma^{-1}(k)} \\
&= \sum_\sigma \mathrm{sgn}(\sigma)\, a_{\sigma^{-1}(j),\,\sigma^{-1}(i),\,\sigma^{-1}(k)} \\
&= \sum_\sigma \mathrm{sgn}(\sigma\,\tau(ij))\, a_{\sigma^{-1}(i),\,\sigma^{-1}(j)\,\sigma^{-1}(k)} \\
&= -\sum_\sigma \mathrm{sgn}(\sigma)\, a_{\sigma^{-1}(i),\,\sigma^{-1}(j)\,\sigma^{-1}(k)} \\
&= \underline{-(a_\wedge)_{ijk}}
\end{aligned}$$

if $d = 2$: $\quad i, j, k \in \{1, 2\}$

$$a^{\wedge}_{1,1,2} = -a_{1,1,2} = 0$$

$\Rightarrow$ all elements $a_{ijk} = 0$

$\boxed{\text{$V = k^d$. the irrep corresponding to a Young diagram is 0 of d is smaller than the number of rows of the Young diagram.}}$

③ $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}$

$$c_{(2,1)} = (e+(12))(e-(13)) = e+(12)-(13)-(132)$$

$$c_{(2,1)} V^{\otimes 3} = \text{span}\{ v_i \otimes v_j \otimes v_k + v_j \otimes v_i \otimes v_k$$
$$- v_k \otimes v_j \otimes v_i - v_k \otimes v_i \otimes v_j \}$$

$$(a_2)_{ijk} = a_{ijk} + a_{jik} - a_{kji} - a_{jki}$$

$i \to k \to j$

$$\left(\begin{array}{l} \sigma : v_i \otimes v_j \otimes v_k \longrightarrow v_{\sigma(i)} \otimes v_{\sigma(j)} \otimes v_{\sigma(k)} \\ a_{ijk} \longrightarrow a_{\sigma^{-1}(i)\sigma^{-1}(j)\sigma^{-1}(k)} \end{array} \right)$$

$i \leftarrow k \leftarrow j$

$$\begin{cases} (a_2)_{ijk} + (a_2)_{jki} + (a_2)_{kij} = 0 & - A \\ (a_2)_{ijk} = -(a_2)_{kji} & - B \end{cases}$$

$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}$: $B \to (a_2)_{ijk} = -(a_2)_{jik}$

2. $c_{(n)} \cdot V^{\otimes n} = \text{Sym}^n V$ projects to the totally symmetric sector. $\Longleftrightarrow$ bosons

$V = k^d = \mathcal{H}$ (single-particle Hilbert space)

$$\dim \text{Sym}^n V = \binom{n+d-1}{n} \qquad \left(\begin{array}{l} v_{i_1} \cdot v_{i_2} \cdots v_{i_n} \\ \quad i_1 \le i_2 \le \cdots \le i_n \end{array} \right)$$

$\Updownarrow$ $(i_n \le d)$

$n=3$ $\frac{1}{6} d(d+1)(d+2)$

$v_{i_1} \cdots v_{i_n}$
$i_1 < i_2 < \cdots < i_n$
$i_n \le d+n-1$

Consider a collection of $\underline{d}$ bosonic oscillators

$$h = \frac{1}{2}\hbar\omega \{a^{+}, a\}$$
$$= \hbar\omega(a^{+}a + \frac{1}{2})$$

$\hbar\omega = 1$. subtract $\frac{1}{2}$

$$H = \sum_{j}^{d} a_j^{+} a_j$$

$(\beta = 1/k_B T)$

Its partition function:

$$Z = \underline{\left(\sum_{n=0}^{\infty} e^{-\beta n}\right)^{d}} = \frac{1}{(1-q)^{d}} \qquad q = e^{-\beta}$$

$$= \sum_{n=0}^{\infty} q^{n} \underline{\dim(\mathrm{Sym}^{n} V)}$$

$\dim(\mathrm{Sym}^{n} V)$ is the degeneracy of eigenstates with total energy n.

2. For fermionic oscillators

$$h = \frac{1}{2}\hbar\omega[a^{+}, a]$$
$$= \hbar\omega(a^{+}a - \frac{1}{2})$$

$$H = \sum_{j}^{d} a_j^{+} a_j$$

$$Z = \left(\sum_{n=0}^{1} e^{-\beta n}\right)^{d} = (1+q)^{d}$$
$$= \sum_{n=0}^{d} q^{n} \underline{\dim(\Lambda^{n} V)}$$

3. $G = U(2) \subset GL(2, \mathbb{C})$

We consider Young diagrams with at most 2 rows.

$$\| \quad T = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 1 & 3 & \cdots & 2k-1 & 2k+1 & \cdots & \cdot & \cdots & 2k+l \\ \hline 2 & 4 & & 2k \\ \cline{1-4} \end{array}$$

$$\underbrace{\qquad\qquad\qquad}_{k} \quad \underbrace{\qquad\qquad\qquad}_{l}$$

The corresponding Young symmetrizer.

$$c_T = P_T \underline{\underline{Q_T}}$$

$$\left(\begin{aligned} &v_{i_1} \wedge v_{i_2} \\ &:= v_{i_1} \otimes v_{i_2} \\ &\quad - v_{i_2} \otimes v_{i_1} \end{aligned} \right)$$

$$c_T \cdot v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_n} \quad (i_m \in \{1, 2\})$$

$$= P_T \, (\underline{v_{i_1} \wedge v_{i_2}}) \otimes (\underline{v_{i_3} \wedge v_{i_4}}) \otimes \cdots \otimes (v_{i_{2k-1}} \wedge v_{i_{2k}})$$

$$Q_T = \prod_{i=1}^{k} e - (2i-1, 2i) \qquad \otimes v_{i_{2k+1}} \otimes \cdots \otimes v_{i_{2k+l}}$$

$$v_{i_{2j-1}} \wedge v_{i_{2j}} \neq 0 \quad \text{iff} \quad i_{2j-1} = 1, \; i_{2j} = 2$$

The non-zero images of c_T is

$$c_T \bigotimes_{j=1}^{n} v_{i_j} = \underline{P_T \left[\bigotimes^{k} (v_1 \wedge v_2) \right]} \otimes v_{i_{2k+1}} \otimes \cdots \otimes v_{i_{2k+l}}$$

$$\propto \underline{\bigotimes_{i}^{k} (v_1 \wedge v_2) \otimes P_{T'} \left(v_{i_{2k+1}} \otimes \cdots \otimes v_{i_{2k+l}} \right)}$$

$\sigma \quad v_1 \wedge v_2 \to v_1 \wedge v_2$

$v_2 \wedge v_1 = -v_1 \wedge v_2$

$$T' : \underbrace{\begin{array}{|c|c|c|c|c|} \hline \; & \; & \; & \; & \; \\ \hline \end{array}}_{l}$$

$V^{\otimes n}$ as rep of $U(2)$.

U(2) acts on $V_1 \wedge V_2$

$$u\cdot(v_1\wedge v_2) = \sum_{i,j} u_{1i}u_{2j}\, v_i\wedge v_j$$

$$= u_{11}u_{22}\, v_1\wedge v_2 + u_{12}u_{21}\, v_2\wedge v_1$$

$$= (u_{11}u_{22} - u_{12}u_{21})\, v_1\wedge v_2$$

$$= (\det u)\, v_1\wedge v_2$$

$$u^{\otimes n}\left(C_T \bigotimes_j^n v_{i_j}\right) = \boxed{(\overset{1}{\cancel{\det u}})^k \bigotimes_i^k (v_1\wedge v_2)} \otimes u^{\otimes \ell} P_{T'}(v_{i_{2k+1}}\otimes\cdots\otimes v_{i_{2k+\ell}})$$

take the subgroup $SU(2) \subset U(2)$ $\quad \det u = 1$

irreps of $SU(2)$ is in one-to-one correspondence with Young diagrams of a single row of ℓ boxes

Dimension of the irrep.

$d=2$

| 1 | 1 | 1 | 1 | 1 | -- | 2 | 2 | 2 | 2 | 2 |

| 1 | 1 | 1 | 2 | -- |

$\dim = \ell+1$

$\mathrm{span}\{ v_{i_1}\otimes \cdots \otimes v_{i_\ell} \}$

$i_1 \le i_2 \le \cdots \le i_\ell$

in physics, $\ell = 2j$ "spin-j representation of $SU(2)$"

$l=0$ scalar / singlet

$l=1$ $(j=\frac{1}{2})$ spin-$\frac{1}{2}$ (dublet) $\boxed{1}$ $\boxed{2}$

$\uparrow$ $\downarrow$

$l=2$ $(j=1)$ triplet $\boxed{1|1}$ $\boxed{1|2}$ $\boxed{2|2}$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$|\uparrow\uparrow\rangle \quad \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \quad |\downarrow\downarrow\rangle$$

Reference Greiner & Müller, Sec. 9.4.

"Quantum mechanics : symmetries"