

Recap.

9

irreps of  $S_n$ :

$$T = \begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & & \\ \hline \end{array} \rightarrow C(T) = P(T) \cup Q(T)$$

$$P(T) = \sum_{f \in P(T)} f$$

$$Q(T) = \sum_{f \in Q(T)} \text{sgn}(f) f$$

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$V = k^d$  is the defining representation  
of  $GL(d, k)$  (and  $U(d)$ )

$V^{\otimes n}$  is a rep of  $S_n$ :

$$\sigma: v_i \otimes v_j \rightarrow v_{\sigma(i)} \otimes v_{\sigma(j)}$$

Schur - Weyl duality:

$$\underline{V^{\otimes n}} \cong \bigoplus_{\lambda} \underline{D_{\lambda}} \otimes \underline{R_{\lambda}}$$

$\lambda$  labels a partition of  $n$ . / Young diagram.

$\Rightarrow$  The representations  $D_{\lambda}$  are irreducible representations of  $GL(d, k)$  (and  $U(d)$ )

All irreps of  $GL(d, k)$  can be obtained by

varying  $n$ .

⇒ Tensors of definite symmetries (obtained via Young symmetrizers) transform as irreps of  $GL(d, K)$ .

Example.  $V^{\otimes 3} = \text{span} \{ v_i \otimes v_j \otimes v_k \}$

$S_3$	$[1^3]$	$3[12]$	$2[123]$
$1^+$	1	1	1
$1^-$	1	-1	1
2	2	0	-1

$\chi([1^3]) = d^3$   
 $\chi([12]) = d^2$   
 $\chi([123]) = d$

$v_i \otimes v_j \otimes v_k$

$$a_{1^+} = \langle \chi_{1^+}, \chi \rangle = \frac{1}{6} (d^3 \cdot 1 + d^2 \cdot 3 + d \cdot 2) = \frac{1}{6} d(d+1)(d+2)$$

$$a_{1^-} = \langle \chi_{1^-}, \chi \rangle = \frac{1}{6} (d^3 - 3d^2 + 2d) = \frac{1}{6} d(d-1)(d-2)$$

$$a_2 = \langle \chi_2, \chi \rangle = \frac{1}{6} (2d^3 - 2d) = \frac{1}{3} d(d+1)(d-1)$$

①  $\boxed{1|2|3}$   $C = PQ = e + (12) + (13) + (23) + (123) + (132)$

$$C \cdot V^{\otimes 3} = \text{span} \left\{ \sum_{\sigma} v_{\sigma(i)} \otimes v_{\sigma(j)} \otimes v_{\sigma(k)} \right\}$$

$$= \text{Sym}^3 V$$

$$t = \sum a_{ijk} v_i \otimes v_j \otimes v_k$$

$$\sigma \cdot t = \sum a_{ijk} v_{\sigma(i)} \otimes v_{\sigma(j)} \otimes v_{\sigma(k)}$$

$$= \sum a_{\sigma^{-1}(i)\sigma^{-1}(j)\sigma^{-1}(k)} v_i \otimes v_j \otimes v_k$$

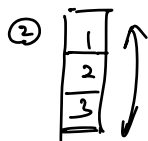
⇒  $(\sigma \cdot a)_{ijk} = a_{\sigma^{-1}(i)\sigma^{-1}(j)\sigma^{-1}(k)}$

$$(a_S)_{ijk} = \sum_{\sigma} a_{\sigma^{-1}(i)\sigma^{-1}(j)\sigma^{-1}(k)} = \sum_{\sigma} a_{\sigma(i)\sigma(j)\sigma(k)}$$

③

$$\Rightarrow (a_s)_{jik} = (a_s)_{ijk}$$

$$(\sigma a_s)_{ijk} = (a_s)_{ijk}$$



$$c = e - (12) - (13) - (23) + (123) + (132)$$

$$(a_\Lambda)_{ijk} = \sum_{\sigma} \text{sgn}(\sigma) a_{\sigma^{-1}(i), \sigma^{-1}(j), \sigma^{-1}(k)}$$

$$(a_\Lambda)_{jik} = (\tau(ij) a_\Lambda)_{ijk}$$

$$= \sum_{\sigma} \tau(ij) \text{sgn}(\sigma) a_{\sigma^{-1}(i), \sigma^{-1}(j), \sigma^{-1}(k)}$$

$$= \sum_{\sigma} \text{sgn}(\sigma) a_{\sigma^{-1}(j), \sigma^{-1}(i), \sigma^{-1}(k)}$$

$$= \sum_{\sigma} \text{sgn}(\sigma \tau(ij)) a_{\sigma^{-1}(i), \sigma^{-1}(j), \sigma^{-1}(k)}$$

$$= - \sum_{\sigma} \text{sgn}(\sigma) a_{\sigma^{-1}(i), \sigma^{-1}(j), \sigma^{-1}(k)}$$

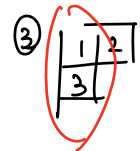
$$= - (a_\Lambda)_{ijk}$$

if  $d=2$ :  $i, j, k \in \{1, 2\}$

$$a_{1,1,2} = -a_{1,1,2} = 0$$

$\Rightarrow$  all elements  $a_{ijk} = 0$

$V = k^d$ . the irrep corresponding to a Young diagram is 0 if  $d$  is smaller than the number of rows of the Young diagram.



$$C_{(2,1)} = (e+(12))(e-(13)) = e+(12) - \underline{(13)} - \underline{(132)}$$

$$C_{(2,1)} V^{\otimes 3} = \text{span} \{ v_i \otimes v_j \otimes v_k + \underline{v_j} \otimes \underline{v_i} \otimes v_k - \underline{v_k} \otimes v_j \otimes \underline{v_i} - \underline{v_k} \otimes v_i \otimes \underline{v_j} \}$$

$$(a_2)_{ijk} = a_{ijk} + \underline{a_{jik}} - \underline{a_{kji}} - \underline{a_{jki}} \quad i \rightarrow k \rightarrow j$$

$$\left( \begin{array}{l} \sigma: v_i \otimes v_j \otimes v_k \rightarrow v_{\sigma(i)} \otimes v_{\sigma(j)} \otimes v_{\sigma(k)} \\ a_{ijk} \rightarrow a_{\sigma^{-1}(i)\sigma^{-1}(j)\sigma^{-1}(k)} \end{array} \right) \quad i \leftarrow k \leftarrow j$$

$$(a_2)_{ijk} + (a_2)_{jki} + (a_2)_{kij} = 0 \quad - A$$

$$\left\{ \begin{array}{l} (a_2)_{ijk} = - (a_2)_{kji} \quad - B \end{array} \right.$$

$$\left[ \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array} \right] : B \rightarrow (a_2)_{ijk} = - (a_2)_{jik}$$

→  $C_{(n)} \cdot V^{\otimes n} = \text{Sym}^n V$  projects to the totally symmetric sector.  $\Leftrightarrow$  bosons

$V = \mathbb{K}^d = \mathcal{H}$  (single-particle Hilbert space)

$$\dim \text{Sym}^n V = \binom{n+d-1}{n} \quad \left( \begin{array}{l} v_{i_1} \cdot v_{i_2} \cdot \dots \cdot v_{i_n} \\ i_1 \leq i_2 \leq \dots \leq i_n \\ \updownarrow \\ (i_n \leq d) \end{array} \right)$$

$$n=3 \quad \frac{1}{6} d(d+1)(d+2)$$

$$\begin{array}{l} v_{i_1} \cdot \dots \cdot v_{i_n} \\ i_1 < i_2 < \dots < i_n \\ i_n \leq d+n-1 \end{array}$$

Consider a collection of  $d$  bosonic oscillators

(5)

$$h = \frac{1}{2} \hbar \omega \{a^\dagger, a\}$$

$$= \hbar \omega (a^\dagger a + \frac{1}{2})$$

$\hbar \omega = 1$ . subtract  $\frac{1}{2}$

$$H = \sum_j^d \alpha_j^\dagger \alpha_j$$

Its partition function:

$$(\beta = 1/k_B T)$$

$$Z = \left( \sum_{n=0}^{\infty} e^{-\beta n} \right)^d = \frac{1}{(1-z)^d} \quad z = e^{-\beta}$$

$$= \sum_{n=0}^{\infty} z^n \underline{\dim(\text{Sym}^n V)}$$

$\dim(\text{Sym}^n V)$  is the degeneracy of eigenstates with total energy  $n$ .

2. For fermionic oscillators

$$h = \frac{1}{2} \hbar \omega [a^\dagger, a]$$

$$= \hbar \omega (a^\dagger a - \frac{1}{2})$$

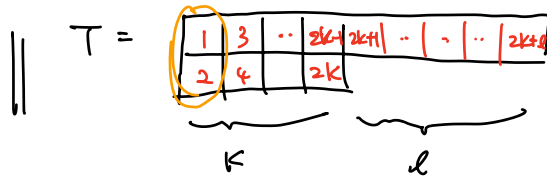
$$H = \sum_j^d \alpha_j^\dagger \alpha_j$$

$$Z = \left( \sum_{n=0}^1 e^{-\beta n} \right)^d = (1+z)^d$$

$$= \sum_{n=0}^d z^n \underline{\dim(\wedge^n V)}$$

3.  $G = U(2) \subset GL(2, \mathbb{C})$

We consider Young diagrams with at most 2 rows.



The corresponding Young symmetrizer.

$$C_T = P_T Q_T =$$

$$C_T \cdot \sigma_{i_1} \otimes \sigma_{i_2} \otimes \dots \otimes \sigma_{i_n} \quad (i_m \in \{1, 2\}) \quad \begin{pmatrix} \sigma_{i_1} \wedge \sigma_{i_2} \\ := \sigma_{i_1} \otimes \sigma_{i_2} \\ - \sigma_{i_2} \otimes \sigma_{i_1} \end{pmatrix}$$

$$= P_T (\sigma_{i_1} \wedge \sigma_{i_2}) \otimes (\sigma_{i_3} \wedge \sigma_{i_4}) \otimes \dots \otimes (\sigma_{i_{2k-1}} \wedge \sigma_{i_{2k}})$$

$$Q_T = \prod_{i=1}^k e^{-\langle 2i-1, 2i \rangle} \otimes \sigma_{i_{2k+1}} \otimes \dots \otimes \sigma_{i_{2k+l}}$$

$$\sigma_{i_{2j-1}} \wedge \sigma_{i_{2j}} \neq 0 \iff i_{2j-1} = 1, i_{2j} = 2$$

The non-zero images of  $C_T$  is

$$C_T \otimes_{j=1}^n \sigma_{i_j} = P_T [\otimes_{i=1}^k (\sigma_{i_{2i-1}} \wedge \sigma_{i_{2i}})] \otimes \sigma_{i_{2k+1}} \otimes \dots \otimes \sigma_{i_{2k+l}}$$

$$\sigma \quad \sigma_{i_1} \wedge \sigma_{i_2} \rightarrow \sigma_{i_1} \wedge \sigma_{i_2} \quad \propto \otimes_{i=1}^k (\sigma_{i_{2i-1}} \wedge \sigma_{i_{2i}}) \otimes P_{T'} (\sigma_{i_{2k+1}} \otimes \dots \otimes \sigma_{i_{2k+l}})$$

$$\sigma_{i_2} \wedge \sigma_{i_1} = -\sigma_{i_1} \wedge \sigma_{i_2}$$

$$T' = \underbrace{\square \square \square \square \square}_{l}$$

$V^{\otimes n}$  as rep of  $U(2)$ .

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$U(2)$  acts on  $v_1 \wedge v_2$

$$\begin{aligned} u \cdot (v_1 \wedge v_2) &= \sum_{i,j} u_{ij} u_{2j} v_i \wedge v_j \\ &= u_{11} u_{22} v_1 \wedge v_2 + u_{12} u_{21} v_2 \wedge v_1 \\ &= (u_{11} u_{22} - u_{12} u_{21}) v_1 \wedge v_2 \\ &= (\det u) v_1 \wedge v_2 \end{aligned}$$

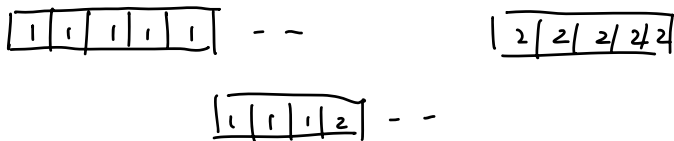
$$u^{\otimes n} (C_T \otimes_j v_{ij}) = (\det u)^k \otimes_i^k (v_1 \wedge v_2) \otimes U^{\otimes l} P_T (v_{i_{l+1}} \otimes \dots \otimes v_{i_{2k+l}})$$

take the subgroup  $SU(2) \subset U(2)$   $\det u = 1$

irreps of  $SU(2)$  is in one-to-one correspondence with Young diagrams of a single row of  $l$  boxes

Dimension of the irrep.

$d=2$



$\dim = l+1$

$\text{span} \{ v_{i_1} \otimes \dots \otimes v_{i_l} \}$   
 $i_1 \leq i_2 \leq \dots \leq i_l$

in physics,  $l=2j$  "spin- $j$  representation of  $SU(2)$ "

$l=0$  scalar / singlet

$l=1$  ( $j=\frac{1}{2}$ ) spin- $\frac{1}{2}$  (doublet)  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$   $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$   
 $\uparrow \downarrow$

$l=2$  ( $j=1$ ) triplet  $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$   $\begin{bmatrix} 1 & 2 \end{bmatrix}$   $\begin{bmatrix} 2 & 2 \end{bmatrix}$   
 $\downarrow \quad \downarrow \quad \downarrow$   
 $|\uparrow\uparrow\rangle \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) |\downarrow\downarrow\rangle$

Reference Greiner & Müller, sec. 9.4.  
"Quantum mechanics: symmetries"