HW. P26. $\underline{\int_G \chi_\mu(g)\chi_\nu(g^{-1}h)\,dg = \frac{\delta_{\mu\nu}}{n_\mu}\chi_\nu(h)}$ (*)

($P_\mu = n_\mu \int_G \overline{\chi_\mu(g)}\; T(g)\,dg$

$P_\mu P_\nu = n_\mu n_\nu \int_{G\times G} \overline{\chi_\mu(g)}\; \underline{\overline{\chi_\nu(h)}}\; T(gh)\,dg\,dh$

$= n_\mu n_\nu \int \underbrace{\overline{\chi_\mu(g)}\;\overline{\chi_\nu(g^{-1}h)}}\; T(h)\,dg\,dh$ $\underline{\quad} = \frac{\delta_{\mu\nu}}{n_\mu}\chi_\nu(h)$

$= \delta_{\mu\nu}\, \underbrace{n_\nu \int \overline{\chi_\nu(h)}\, T(h)\,dh}_{P_\nu}$

LHS $= \int_G \sum_i M^\mu_{ii}(g) \sum_j \left(\sum_k \underline{M^\nu_{jk}(g^{-1})}\; \underline{M^\nu_{kj}(h)}\right] dg$

$\boxed{\chi_\mu(g^{-1}h) = \chi_\mu(g^{-1})\chi_\mu(h)}$ WRONG !

μ: V^2 $\underset{2}{\chi(e)} = \underset{0}{\chi((12))}\,\underset{0}{\chi((12))}$

Recap. S_n. irreps

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}$$

standard tableaux: $\longrightarrow$ increasing integers.

$\downarrow$

1. $c = PQ$

$$P = \sum_{p \in R(T)} p \qquad R(T) = S_3$$
$$C(T) = \{e, (14)\}$$

$$Q = \sum_{q \in C(T)} \operatorname{sgn}(q)\, q$$

$c(T)$,

① $c^2 = \lambda c$ ($\lambda > 0$ integers)

② $c(T)\, c(T') = 0$ if $T \neq T'$.

c essentially idempotent

$\left\{ \begin{array}{l} \propto \text{ projectors onto irreps} \\ \text{Young symmetrizer} \end{array} \right.$

2. dim of irrep corresponding to a Young diagram $=$ # of standard tableaux.

Example. S_4

$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline\end{array}$ trivial dim $=1$

$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline\end{array}$ sgn $=1$

irreps of S_n (cont.)

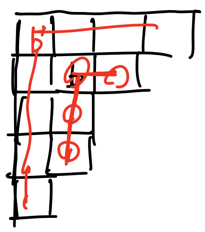
For a given T,

$$c(T)^2 = \lambda(T)\, c(T)$$

$$\lambda(T) = \frac{n!}{f} \qquad f : \text{dim of irrep.}$$

$$f = \frac{n!}{\prod_b h(b)} \qquad \text{"hook length formula"}$$

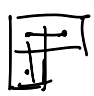
$h(b)$: hook length.



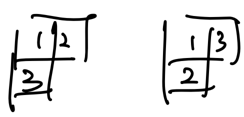
$h(b) = 4$

$h(b') = 8$

S_3 :

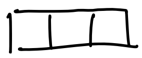
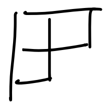
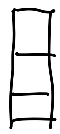


$$f = \frac{3!}{3} = 2$$



Example S_3:

①

	diagrams	standard tableau(x)
trivial:		$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}$
standard:		$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ $\quad$ $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$
sgn:		$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$

② trivial. $P = \sum_{p \in R(T)} p = e + (12) + (13) + (23) + (123) + (132)$

$Q = e$

$(\tilde{c}^2 = \hat{c})$ $\qquad \lambda = \frac{n!}{f} = 6$

$$\underline{\tilde{c} = \frac{1}{\lambda} c = \frac{1}{6}\left(e + (12) + (13) + (23) + (123) + (132)\right)}$$

$\forall \phi \in S_3 \qquad \underline{\phi \tilde{c}} = \tilde{c}$

$$\underline{\underline{R_{S_3}}} \cdot \tilde{c} = \tilde{c}$$

sgn: $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$ $\qquad P = e$

$Q = e - (12) - (13) - (23) + (123) + (132)$

$$\tilde{c} = \frac{1}{6} Q$$

$$\phi \tilde{c} = \mathrm{sgn}(\phi)\, \tilde{c} \qquad (\phi \in S_3)$$

$\{ R_{S_3} \cdot \tilde{c} \}$ 1D sgn

standard :

$$\underset{T_1}{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}} \qquad \underset{T_2}{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}} \qquad f = \frac{3!}{3} = 2$$

$$\lambda = 3$$

T_1 : $P_1 = e + (12)$

$(12)(13) = (132)$

$Q_1 = e - (13)$

$$\begin{cases} \tilde{C}_1 = \frac{2}{6} P_1 \cdot Q = \frac{1}{3}(e - (13) + (12) - (132)) \\ \tilde{C}_2 = \frac{1}{3}(e - (12) + (13) - (123)) \end{cases}$$

$$\begin{cases} \tilde{C}_i \tilde{C}_i = \tilde{C}_i \\ \tilde{C}_1 \tilde{C}_2 = 0 \end{cases}$$

check !

$\boxed{R_{S_3} \cdot \tilde{C}_1 :}$

$(12)(132) = (13)(2)$

$e \cdot \tilde{C}_1 = \tilde{C}_1 = v_1$

$(12) \cdot \tilde{C}_1 = \frac{1}{3}((12) - (132) + e - (13))$

$= \tilde{C}_1$

$(13)(132) = (1)(23)$

$(13) \cdot \tilde{C}_1 = \frac{1}{3}((13) - e + (123) - (23))$

$=: v_2$

$(23) \cdot \tilde{C}_1 = -v_1 - v_2$

$(123) \cdot \tilde{C}_1 = v_2$

$(132) \cdot \tilde{C}_1 = -v_1 - v_2$

Matrix rep. of $V = \mathrm{span}\{v_1, v_2\}$

$$\begin{cases} (12) \cdot v_1 = v_1 \\ (12) \cdot v_2 = (12)((13) \cdot v_1) = (132) \cdot v_1 = -v_1 - v_2 \end{cases}$$

$$M[(12)] = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \qquad \chi_2(12) = 0$$

$$\begin{cases} (13)\cdot v_1 = v_2 \\ (13)\, v_2 = v_1 \end{cases}$$

$$M[(13)] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \underline{\chi_2(13) = 0}$$

$$M[(23)] = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} \qquad \chi_2(23) = 0$$

$$M[(123)] = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \qquad \underline{\chi_2(123) = -1}$$

---

recall.

class operators $\qquad \hat{C}_i = \underline{\sum \lambda^i_\mu P^\mu}$

$$\hat{L} = \begin{pmatrix} y^1 & y^2 & y^3 \\ 3y^2 & y^1+2y^3 & 3y^2 \\ 2y^3 & 2y^2 & y^1+y^3 \end{pmatrix}$$

$$\lambda_1 = y^1 + 3y^2 + 2y^3$$
$$\lambda_2 = y^1 - 3y^2 + 2y^3$$
$$\lambda_3 = y^1 + 0 - y^3$$

$$\begin{cases} \hat{C}_1 = \underline{P_{\mu_1} + P_{\mu_2} + P_{\mu_3}} \\ \hat{C}_2 = \underline{3P^{\mu_1} - 3P^{\mu_2}} \\ \hat{C}_3 = \underline{2P^{\mu_1} + 2P^{\mu_2} - P^{\mu_3}} \end{cases}$$

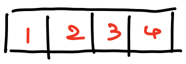
$$\tilde{C}\left(\boxed{1\,|\,2\,|\,3}\right) = P^{\mu_1} = \frac{1}{6}\left(\hat{C}_1 + \hat{C}_2 + \hat{C}_3\right)$$

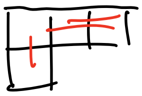
$$\tilde{C}\left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}\right) = P^{\mu_2} = \frac{1}{6}\left(\hat{C}_1 - \hat{C}_2 + \hat{C}_3\right)$$

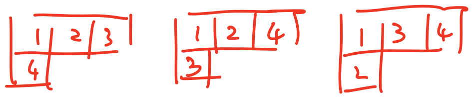
$$\tilde{C}\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\right) + \tilde{C}\left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\right) = \underline{P^{\mu_3}} = \frac{1}{3}\left(2\hat{C}_1 - \hat{C}_3\right)$$
$$= \frac{1}{3}\left(2e - (123) - (132)\right)$$

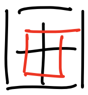
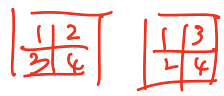
Example : Character table of S_4.

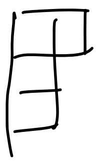
1. Conjugacy classes ?   2. irreps ? = # conj. class.

(4)  $f = \frac{4!}{\prod_b h(b)} = 1$ 1

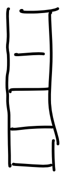
(3)(1)  $f = \frac{4!}{4 \times 2} = 3$ 3



$(2)^2$  $f = \frac{4!}{3 \times 2 \times 2} = 2$    2

$(2)(1)^2$ 

1 2 / 3 / 4   1 3 / 2 / 4   1 4 / 2 / 3   3

$(1)^4$ 

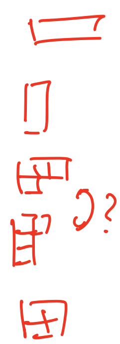
1 / 2 / 3 / 4   1

$|G| = \sum_\mu n_\mu^2$

$1 + 3^2 + 2^2 + 3^2 + 1 = 24 = 4!$

	E	$\binom{4}{2}=6$ 6 [(12)]	$\binom{4}{2}/2$ 3 [(12)(34)]	$\binom{4}{3}\cdot 2$ 8 [(123)]	6 [(1234)]
V^+	1	1	1	1	1
V^-	1	-1	1	1	-1
$V^\perp$	3	1	-1	0	-1
$V^- \otimes V^\perp$	3	-1	-1	0	1
V^2	2	0	2	-1	0

V^{R^n}	4	2	0	1	0

 ?

$S_n \quad \{e_i\} \quad \mathbb{R}^n \qquad L = \Sigma e_i$

$L^\perp =$

$\underline{V}^{\mathbb{R}^n} \cong \underline{V^+ \oplus V^\perp}$

$\langle \chi^\Gamma, \chi^\Gamma \rangle = 1 \Leftrightarrow$ irrep.

S_3 HW

$V^+ \otimes V^\mu \cong V^\mu$

$\underline{V^- \otimes V^2 \cong V^2}$

$V^- \otimes V^- \cong V^+$

8.14. Schur-Weyl duality : irreps of $GL(d, K)$

$V^{\otimes 2}$ as a representation of S_2: $(V = K^d, \ K = \mathbb{R}, \mathbb{C})$

$\sigma : v_1 \otimes v_2 \longmapsto v_2 \otimes v_1$

$V \otimes V \cong \underline{D^{1+} \otimes \mathbb{1}^+} \oplus D^{1-} \otimes \mathbb{1}^-$

$\begin{cases} \dim D^{1+} = \frac{d(d+1)}{2} \\ \dim D^{1-} = \frac{d(d-1)}{2} \end{cases}$

$D^{1+} \otimes \mathbb{1}^+ = \text{span}\{ v_i \otimes v_j + v_j \otimes v_i \}$

$=: \text{span}\{ v_i \cdot v_j \, , \ i \leq j \} = \text{Sym}^2 V$

$D^{1-} \otimes \mathbb{1}^- = \text{span}\{ v_i \otimes v_j - v_j \otimes v_i \}$

$=: \text{span}\{ v_i \wedge v_j \, , \ i < j \} = \Lambda^2 V$

$\boxed{1}\boxed{2}$ $\quad c = e + (12) \qquad v_i \otimes v_j \longmapsto v_i \otimes v_j + v_j \otimes v_i$

$$c \cdot V^{\otimes 2} = \operatorname{span}\{ v_i \otimes v_j + v_j \otimes v_i \} \simeq \operatorname{Sym}^2 V$$

$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}$ $\quad c = e - (12)$

$$c \cdot V^{\otimes 2} = \operatorname{span}\{ v_i \otimes v_j - v_j \otimes v_i \} \simeq \Lambda^2 V$$

$$\pi : V \otimes V \longrightarrow \operatorname{Sym}^2 V$$

$$\ker(\pi) = \{ v_i \otimes v_j - v_j \otimes v_i \}$$

$$\pi : V \otimes V \longrightarrow \Lambda^2 V$$

$$\ker(\pi) = \{ v_i \otimes v_j + v_j \otimes v_i \}$$

Any elemets $\in V^{\otimes 2}$. can be given by a rank-2 tensor

$$t = \sum_{ij} a_{ij}\, v_i \otimes v_j$$

Then the action of S_2

$$\sigma \cdot t = \sum_{ij} a_{ij}\, v_{\sigma(i)} \otimes v_{\sigma(j)} = \sum_{ij} a_{\sigma^{-1}(i)\sigma^{-1}(j)}\, v_i \otimes v_j$$

defines an action on the tensor:

$$(\sigma \cdot a)_{ij} = a_{\sigma^{-1}(i)\sigma^{-1}(j)} \qquad (a \in K^{d^2})$$

V a rep. of group G. $V \otimes V$ is a rep:

$$T(g)^{\otimes 2}(v_1 \otimes v_2) = T(g)v_1 \otimes T(g)v_2$$

$$T(g)\cdot t = \sum_{ij} a_{ij} [\underline{T(g)v_i} \otimes T(g)v_j]$$

$$= \sum_{\substack{ij \\ kl}} a_{ij} M(g)_{ki} M(g)_{lj} v_k \otimes v_l$$

defines an action on $\underline{a}$.

$$(g\cdot a)_{kl} = \sum_{ij} M(g)_{ki} M(g)_{lj} a_{ij}$$

The action of G and S_2 commutes

(show): $[\sigma\cdot(g\cdot a)]_{ij} = [g(\sigma a)]_{ij}$

$(a_s)_{ij} = a_{ij} + a_{ji}$ $\quad$ $\dim a_s = \frac{d(d+1)}{2}$

$(a_A)_{ij} = a_{ij} - a_{ji}$ $\quad$ $\dim a_A = \frac{d(d-1)}{2}$

$\Rightarrow$ The degeneracy space of different irreps of S_2 is also a rep of G.

<u>Schur–Weyl duality theorem</u>: (Fulton & Harris for proofs)

$$V^{\otimes n} \cong \bigoplus_\lambda D_\lambda \otimes R_\lambda$$

R_λ are the irreps of S_n

$D_\lambda = \mathrm{Hom}_{S_n}(R_\lambda, V^{\otimes n})$ the degeneracy space.

The representations D_λ are irreducible representations of $GL(d, K)$