Recap. 8.12. Group algebra.

Refs.

① Fulton & Harris. Representation theory. (GTM 129)

Sec. 3.4.

☆ ② Miller. "Symmetry groups and their applications".

Chap 3

Chap 4 symmetric group rep.

☆ ③ 陈金全. 第二章 群表示基础

"群元既是算符, 又是基矢".

$R_G = \mathrm{span}\{g,\ g \in G\}$ $\qquad G \to GL(R_G)$

$$\delta_{C_i} = \begin{cases} 1 & g \in C_i \\ 0 & \text{otherwise} \end{cases}$$

$$\delta_{C_i} = \sum_{g \in G} \delta_{C_i}(g) \cdot g = \sum_{g \in C_i} g =: \hat{C}_i$$

class function $\qquad\qquad$ class operator.

$\forall g \in G$, $g C_i = C_i g$

$$[L(h)\varphi_j^{\mu}](g) = \varphi_j^{\mu}(h^{-1} g) \qquad \varphi \in R_G$$

$$= \sum_i T(h^{-1})_{ij} \varphi_i^{\mu}(g)$$

For nondegenerate case

$$\underline{L(h)}\,\underline{\varphi^{\mu}} = \alpha\, \varphi^{\mu} \qquad \alpha \in \mathbb{C}$$

$$L(h)\left(\underline{\hat{C}_i\, \varphi^{\mu}}\right) = \hat{C}_i\, \underline{L(h)\, \varphi^{\mu}} = \alpha \left(\hat{C}_i\, \varphi^{\mu}\right)$$

$$\underline{\hat{C}_i} \cdot \varphi^{\mu} = \beta\, \underline{\varphi^{\mu}} \qquad \beta \in \mathbb{C}$$

$$\underline{\hat{C}_i} \cdot \underline{\hat{C}_j} = \underline{\sum_k M_{ij}^{k}}\ \hat{C}_k \qquad \langle C_i, C_j \rangle \propto \delta_{ij}$$

$$\underline{\hat{C}_i} \cdot \delta_{C_j} = [M^i]_{jk}\, \delta_{C_k}$$

$$\langle \delta_{C_k} | \hat{C}_i | \delta_{C_j} \rangle = [M^i]_{jk}$$

$$C_i = \begin{pmatrix} \lambda^{\mu} & & & \\ & \lambda^{\mu} & & \\ & & \ddots & \\ & & & \lambda^{\mu_r} \end{pmatrix}$$

o $\quad C_i \cong \oplus\, C_i^{\mu} \cong \oplus\, \underline{\lambda_i\, \mathbb{1}_{V^{\mu}}}$

$$\mathrm{Tr}_{V^{\mu}}(C_i) = \mathrm{Tr}_{V^{\mu}} \underline{\sum_{g \in C_i} g} = m_i\, \chi^{\mu}([C_i]) = \lambda_i \cdot n_{\mu}$$

$n_{\mu} = \dim V^{\mu}$

eigenvalues of C_i $\quad \boxed{\lambda_i^{\mu}} = \dfrac{m_i}{n_{\mu}}\, \underline{\chi_{\mu}}([C_i])$ $\rightarrow$ known for a given group.

$$\frac{1}{|G|} \sum_{C_i} m_i\, \chi_{\mu}(C_i)\, \overline{\chi_{\nu}(C_i)} = \delta_{\mu\nu}$$

$\hookrightarrow$

$$\chi_{\mu} = \frac{\lambda_i^{\mu}}{\sqrt{\langle \lambda_i^{\mu}, \lambda_i^{\mu} \rangle}} \qquad \langle \lambda_i^{\mu}, \lambda_i^{\nu} \rangle = \frac{1}{|G|} \sum_{C_i} m_i\, \lambda_i^{\mu}\, \overline{\lambda_i^{\nu}}$$

$$n_{\mu} = \frac{m_i}{\sqrt{\langle \lambda_i^{\mu}, \lambda_i^{\mu} \rangle}}$$

8.12.1. Construction of character table (cont.) ①

$$\hat{C}_i = \sum_{\mu=1}^{r} \lambda_i^{\mu} P^{\mu} \qquad (P^{\mu} P^{\nu} = \delta_{\mu\nu} P^{\mu})$$

$$\cong \oplus \lambda_i^{\mu} \mathbb{1}_{V_\mu}$$

$$C_i^{(\mu)} = \frac{m_i}{n_\mu} \chi_\mu([C_i]) \mathbb{1}_{V_\mu}$$

$$\underline{\hat{C}_i \cdot \hat{C}_j = \sum_k C_{ij}^k \hat{C}_k}$$ write out on a specific irrep μ.

$$\left(\frac{m_i}{n_\mu} \chi_\mu([C_i]) \right) \left(\frac{m_j}{n_\mu} \chi_\mu([C_j]) \right) = \sum_k C_{ij}^k \left(\frac{m_k}{n_\mu} \chi_\mu([C_k]) \right)$$

Define. $\varphi_i = m_i \chi_\mu([C_i])$

$$LHS = \frac{1}{n_\mu^2} \varphi_i \varphi_j$$

$$RHS = \frac{1}{n_\mu} \sum_k C_{ij}^k \varphi_k$$

We want to diagonalize all C_i's simultaneously.

introduce auxillary variables $\underline{\underline{y^i}}$.

$$\sum LHS = \underline{\underline{\frac{1}{n_\mu^2} \sum_i (\varphi_i y^i) \varphi_j}}$$

$$\sum RHS = \frac{1}{n_\mu} \sum_{ik} (C_{ij}^k y^i) \varphi_j = \underline{\frac{1}{n_\mu} \sum_k L_j^k \varphi_k}$$

$$L_j^k \equiv \sum_i C_{ij}^k y^i$$

$$\Rightarrow \underline{\sum_{k=1}^{r} L_j^k \varphi_k = \lambda \varphi_j}$$

$$\lambda \equiv \frac{1}{n_\mu} \sum_{i=1}^{r} \varphi_i y^i$$

Solve the above eigen problem. we obtain a set of eigenvalues $\{\lambda_\mu\}$

$$\boxed{\lambda_\mu = \frac{1}{n_\mu}\sum_{i=1}^{r} m_i \chi_\mu([C_i])\, y^i} \quad (*)$$

$$\frac{1}{|G|}\sum_{C_i} m_i \chi_\mu(C_i)\overline{\chi_\nu(C_i)} = \delta_{\mu\nu}$$

$$\xRightarrow{\mu=\nu} \sum_{i=1}^{r} m_i |\chi_\mu([C_i])|^2 = |G|$$

$$|G| = |\chi_\mu([C_e])|^2 \sum m_i \left|\frac{\chi_\mu([C_i])}{\chi_\mu([C_e])}\right|^2$$

$$= n_\mu^2 \sum_{i=1}^{r} m_i \left|\frac{\chi_\mu([C_i])}{n_\mu}\right|^2$$

$$\Rightarrow n_\mu = \left[\frac{|G|}{\sum_{i=1}^{r} m_i \left|\frac{\chi_\mu([C_i])}{n_\mu}\right|^2}\right]^{\frac{1}{2}}$$

$\to$ obtain from λ

Example S_3

S_3: E; $(12)\ (13)\ (23)$; $(123), (132)$

① class operator:

$$C_1 = E$$

$$C_2 = (12) + (13) + (23)$$

$$C_3 = (123) + (132)$$

2 class multiplication table

	C_1	C_2	C_3
C_1	C_1	C_2	C_3
C_2	C_2	$3C_1+2C_3$	$2C_2$
C_3	C_3	$2C_2$	$2C_1+C_3$

symmetric along diagonal.

$(12)(13) = (132)$

$(12)(23) = (1)(23)$

$$C_2 \cdot C_2 = \left((12)+(13)+(23) \right)^2$$

$$= \underbrace{(12)(12)}_{e} + \underbrace{(12)(13)}_{(132)} + \underbrace{(12)(23)}_{(231)=(123)}$$

$$+ \underbrace{(13)(12)}_{(123)} + \underbrace{(13)(13)}_{e} + \underbrace{(13)(23)}_{(213)=(132)}$$

$$+ \underbrace{(23)(12)}_{(132)} + \underbrace{(23)(13)}_{(123)} + \underbrace{(23)(23)}_{e}$$

$$= 3e + 3\left[(123)+(132)\right]$$

$$= 3C_1 + 3C_3$$

3 $L_{jk} = \sum_i C_{ij}^k \, y^i$

	C_1	C_2	C_3
C_1	C_1	C_2	C_3
C_2	C_2	$3C_1+2C_3$	$2C_2$
C_3	C_3	$2C_2$	$2C_1+C_3$

$$L_{11} = C_{11}^1 y^1 + C_{21}^1 y^2 + C_{31}^1 y^3$$

$$= y^1 + 0 + 0$$

$$L_{12} = C_{11}^2 y^1 + C_{21}^2 y^2 + C_{31}^2 y^3 = 0 + y^2 + 0$$

$$\vdots$$

$$\hat{L} = \begin{pmatrix} y^1 & y^2 & y^3 \\ 3y^2 & y^1+2y^3 & 3y^2 \\ 2y^3 & 2y^2 & y^1+y^3 \end{pmatrix}$$

$$
\begin{array}{c} C_1 \\ = \underline{\underline{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}} \end{array} y^1 + \underline{\underline{\begin{pmatrix} 0 & 1 & 0 \\ 3 & 0 & 3 \\ 0 & 2 & 0 \end{pmatrix}}} y^2
$$

$$
+ \underline{\underline{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix}}} y^3
$$

④

Diagonalize $\hat{C}$. eigenvalues.

$$
\begin{cases} \lambda_a = y^1 + 3y^2 + 2y^3 \\ \lambda_b = y^1 - 3y^2 + 2y^3 \\ \lambda_c = y^1 + 0y^2 - y^3 \end{cases}
$$

$$
m_i := (1, \; 3, \; 2)
$$

$$
\lambda_\mu = \underline{\underline{\sum_{i=1}^{r} \frac{m_i \chi_\mu([C_i])}{n_\mu}}} y^i
$$

$$
n_\mu = \left[\frac{|G|}{\sum_{i=1}^{r} m_i \left| \frac{\chi_\mu(C_i)}{n_\mu} \right|^2} \right]^{\frac{1}{2}}
$$

V^+ $\quad x_a = n_a (1, \; 1, \; 1)$ $\qquad n_a = \left(\frac{6}{1+3+2} \right)^{\frac{1}{2}} = 1$

V^- $\quad x_b = n_b (1, \; -1, \; 1)$ $\qquad n_b = 1$

$\quad x_c = n_c (1, \; 0, \; -\frac{1}{2})$ $\qquad n_c = \left(\frac{6}{1+3 \cdot 0+2 \cdot \frac{1}{4}} \right)^{\frac{1}{2}} = 2$

V^2 $\quad x_c = (2, \; 0, \; -1)$

§13 Representation of S_n (Miller, book Chap 4)

see also 陈金全

contains all proofs of the statements below.

Basics of S_n:

$(i_1, i_2, \cdots i_r) \sim (j_1, j_2, \cdots j_r)$

r-cycles are conjugate

S_n irreps are defined by vectors

$\vec{\ell} = (\ell_1, \ell_2 \cdots \ell_n)$

ℓ_i the number of i-cycles

conj. classes $\Leftrightarrow$ Young diagrams.

Continue of the group algebra perspective

finding irreps = finding (primitive) idempotents.

For 1D irreps:

① $\underline{c} = \frac{1}{n!} \sum_{s \in S_n} s_n$ $\quad \underline{cs} = \underline{sc} = c \quad \underline{c^2 = c}$.

$(\forall s \in S_n)$

The subspace $\{\lambda c\}$ is an irrep.

$L(s) \cdot c = sc = c$

trivial irrep

② $\underline{C} = \frac{1}{n!} \sum_{s \in S_n} \mathrm{sgn}(s) \cdot s$

$$Cs = sC = \mathrm{sgn}(s) \cdot C \quad \forall s \in S_n$$

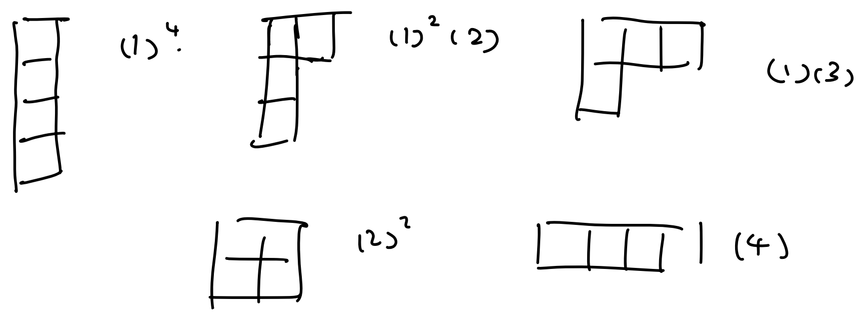
$$L(s) \cdot C = \mathrm{sgn}(s) \cdot C.$$

sgn irrep

How to find projectors / idempotents onto other irreps ?

$\Rightarrow$ use Young diagrams & Young tableaux.

$\underline{S_4}$.



Young tableaux:

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&&\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&&\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 4&1&2\\\hline 3&&\\\cline{1-1}\end{array}$$

$n!$ tableaux for a diagram

standard tableau: integers increase within row & column.

Given a tableau T. we define two sets of permutations $R(T)$, $C(T)$

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & & \\ \cline{1-1}\end{array} \qquad \begin{aligned} R(T) &= \{e,\ (12),\ (13),\ (23),\ (123),\ (132)\} \\ C(T) &= \{e,\ (14)\} \end{aligned}$$

$$R(T) \cap C(T) = \{\underline{e}\}$$

$\Big($ $p \in R(T)$, $q \in C(T)$ $\quad \underline{\underline{pq}}$ unique.

p' $\quad q'$

$p'q' = pq \iff \underline{\underline{q(q')^{-1}}} = \underline{\underline{p^{-1} \cdot p'}} = e \implies p = p',\ q = q'$ $\Big)$ (*)

Then we construct two elements of $R_{S_n} =: R_n$

$$P = \sum_{p \in R(T)} p \qquad Q = \sum_{q \in C(T)} \epsilon(q) \cdot q \qquad \left(\epsilon(q) = \mathrm{sgn}(q) \in \{\pm 1\} \right)$$

$$\underline{\underline{c}} = PQ = \sum_{\substack{p \in R(T) \\ q \in C(T)}}{}' \epsilon(q)\, \underline{\underline{pq}} \qquad \left(\overset{(*)}{\neq} 0 \right)$$

<u>Theorem 1</u>. $c = PQ$ corresponding to a tableau T is <u>essentially idempotent</u>

The invariant subspace $R_n c$ $\left(= \{ g c,\ \forall g \in R_n \} \right)$ yields an irrep of S_n.

That is to say :

① $c^2 = \lambda c$ ($\lambda > 0$ integers)

($\tilde{c} = \lambda^{-1} c$ idempotent)

$P^{\mu} P^{\nu} = P^{\mu} \delta_{\mu\nu}$

② $c \cdot c' = 0$ ($\forall c' \neq c$, T' a different tableau)

Theorem 2. The dimension f of the irrep corresponds to a diagram is the number of standard tableaux $\{T_i,\ i=1, \cdots f\}$

Example

S_4 $\begin{cases} \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array} & \text{trivial} \quad f = 1 \\ \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array} & \text{sgn} \quad f = 1 \end{cases}$

S_3 $\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}$ $\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}$ $f = 2$ standard irrep.