HW. $G = (\mathbb{R}^+, \times)$

$$\chi(a) = e^{ikf(a)} \qquad f: \mathbb{R} \to \mathbb{R} \quad \underline{k \in \mathbb{R}}$$

$$\chi(a \cdot_G b) = \chi(a) \cdot \chi(b) \qquad e^{ikf(ab)} = e^{ikf(a)} \cdot e^{ikf(b)}$$

$$f(ab) = f(a) + f(b)$$

$$f(a) \propto \ln a$$

$$\chi_k(a) = e^{ik\ln(a)}$$

$$(\chi_{k_1} \cdot \chi_{k_2})(a) = \chi_{k_1}(a) \cdot \chi_{k_2}(a) = \chi_{k_1+k_2}(a)$$

$$\hat{G} = (\mathbb{R}, +) \qquad \hat{G} = (\mathbb{R}, +)$$

2. $\underline{(\mathbb{R}^+, \times) \cong (\mathbb{R}, +)}$

$$\varphi : (\mathbb{R}, +) \longrightarrow (\mathbb{R}^+, \times)$$

$$a \longmapsto e^a$$

$$a+b \longmapsto e^a \cdot e^b = e^{a+b}$$

$$e^a = 1 \quad \text{iff} \quad a = 0$$

<u>Recap</u>. Summary of key results.

① unitary rep. of <u>compact</u> G.

$$\langle M^{\mu_1}_{i_1 j_1}, M^{\mu_2}_{i_2 j_2} \rangle = \frac{1}{n_\mu} \delta^{\mu_1 \mu_2} \delta_{i_1 i_2} \delta_{j_1 j_2}$$

complete, orthogonal basis of $L^2(G)$.

② (Peter-Weyl) $L^2(G) \underset{\mathcal{U}}{\cong} \bigoplus_\mu \mathrm{End}(V^\mu)$

$$\iota : \bigoplus_\mu \mathrm{End}(V^\mu) \longrightarrow L^2(G)$$

$$\underline{\underline{\bigoplus_\mu S_\mu}} \longmapsto \sum_\mu \varphi_{S_\mu}$$

$$\underline{\varphi_{S_\mu} := \mathrm{Tr}_{V_\mu}(S\,T(g^{-1}))}$$

$\hookrightarrow$ finite G: $\boxed{|G| = \sum_\mu n_\mu^2}$

$(n_\mu = \dim V^\mu)$

③ characters.

$$\int_G \overline{\chi^{(\mu)}(g)}\, \chi^{(\nu)}(g)\, dg = \delta_{\mu\nu}$$

ON basis of $L^2(G)^{\text{class}}$.

④ $V \underset{\mathcal{U}}{\cong} \bigoplus_\mu a_\mu V^{(\mu)}$

$$a_\mu = \int_G \overline{\chi^{(\mu)}}(g)\, \chi_V(g)\, dg = \langle \chi^{(\mu)}, \chi_V \rangle$$

reg. rep. $a_\mu = \langle \chi^\mu, \underline{\chi} \rangle = \frac{1}{|G|} (\dim n_\mu) \cdot |G|$

$$\underline{= \dim n_\mu}$$

⑤ # irreps = # conj. class.

		C_i	- - - -
irreps	Γ_1		
	Γ_2		
	$\vdots$		

rows: $\frac{1}{|G|}\sum_{\{C_i\}} |C_i| \, \chi_\mu(C_i) \, \overline{\chi_\nu(C_i)} = \delta_{\mu\nu}$

columns: $\sum_{\mu} \overline{\chi_\mu(C_i)} \, \chi_\mu(C_j) = \frac{|G|}{m_i} \delta_{ij}$

8.12 Group algebra (of finite groups).

representation : $G \longrightarrow GL(V)$

Introduce a new vector space R_G (group ring)

Let G be a finite group of order n.

Define n-dim vector space R_G with basis $\{g.\ g \in G\}$

$$x = \sum_{g\in G} x(g)\cdot g \qquad x \in R_G \qquad x(g) \in \mathbb{C}.$$

$x = y$ iff $\forall g \in G.\ x(g) = y(g)$

$$x + y = \sum_{g\in G} x(g)\cdot g + \sum_{g\in G} y(g)\cdot g = \sum (x(g)+y(g))g$$

$$\alpha x = \sum_{g\in G} \alpha x(g)\cdot g \qquad \alpha \in \mathbb{C}.$$

$$0 = \sum_{g\in G} 0\cdot g$$

$$xy = (\sum x(g)\cdot g)(\sum y(g)\cdot g) = \sum_{g,h} x(g)y(h)\,gh$$

$$\overset{gh\to k}{=} \sum_k \Big(\sum_g x(g)\,y(g^{-1}k)\Big)\cdot k = \sum_k xy(k)\cdot k$$

$$xy(k) = \sum_g x(g)\,y(g^{-1}k) \quad \text{convolution product}$$

$$\left((f*g)(t) = \int f(\tau)\, g(t-\tau)\, d\tau \right)$$

$\Rightarrow R_G$ is a group ring / group algebra $\underline{K[G]}$

(commutative +, distributive × etc.) $\mathbb{C}[G]$

Review of basic ideas of rep. theory:

Regular representation: $G \times G$

$$(g_1, g_2) \longmapsto L(g_1) R(g_2^{-1})$$

$$(g_1, g_2) x = g_1 x g_2^{-1} \quad \left(\begin{array}{l} g_i \in G \\ x \in R_G \end{array} \right)$$

$$L \,\&\, R : G \longrightarrow GL(R_G)$$

$\hookrightarrow$ restrict to subgroups $G \times \{1\}$. or $\{1\} \times G$.

LRR: $L(g) \cdot x = g x$

RRR: $R(g) x = x g^{-1}$

$$L(h) \cdot x = L(h) \cdot \sum_g x(g) \cdot g = \sum_g x(g)(hg) = \sum_g x(h^{-1}g) \cdot g$$

$$\equiv \sum_g [L(h) \cdot x](g) \cdot g$$

$[L(h) \cdot x](g) = x(h^{-1}g)$ (recall induced action on functions on G.)

$[R(h) \cdot x](g) = x(g \cdot h)$

Define inner product

$$\langle x, y\rangle = \int_G \overline{x(g)}\, y(g)\, dg$$

$$\overset{\text{finite}}{=} \frac{1}{|G|} \sum_g \overline{x(g)}\, y(g)$$

$\Rightarrow \langle L(h)x, L(h)y\rangle = \langle x, y\rangle$

L, R are unitary reps.

View x also as functions on G. $x: G \to \mathbb{C}$

$g \mapsto x(g)$

$\underline{h} = \sum_g h(g)\cdot g = 1\cdot h \quad \Rightarrow \quad h(g) = \begin{cases} 1 & g = h \\ 0 & \text{otherwise} \end{cases}$

$\parallel$

(recover $\underline{\delta_h}$ from before)

$\underline{\delta_h \cdot \delta_g} = \sum_k \left(\sum_l \delta_h(l)\, \delta_g(l^{-1}\cdot k)\right)\cdot k = 1\cdot(hg) = \underline{\delta_{hg}}$

$l = h.$

$l^{-1}\cdot k = g \quad k = hg$

see h as left action: $\underline{L(h)\,\delta_g(g')} = \delta_g(h^{-1}\cdot g') = \delta_{hg}(g')$

$\underline{L(h)\delta_g = \delta_{hg}}$

group elements can be viewed both as operators and vectors on R_G

Also. expand the class function on R_G:

$$\delta_{C_i}(g) = \begin{cases} 1 & g \in C_i \\ 0 & \text{otherwise} \end{cases}$$

$$\delta_{C_i} = \sum_{g\in G} \delta_{C_i}(g)\cdot g = \sum_{g\in C_i} g$$

(or view as class operators C_i)

$\forall h \in G$: $h C_i h^{-1} = \sum_{g\in C_i} h g h^{-1} = \sum_{g'\in C_i} g' = C_i$. C_i commutes with $\forall h \in G$

Projectors and invariant subspaces:

$V = \bigoplus_i W^i$ invariant subspace.

Suppose. $V = W \oplus W^\perp$

Define projector P onto W.

$\forall x \in V$. $x = w + w^\perp$ $w \in W$, $w^\perp \in W^\perp$

then $Px = w$ $\quad gw \in W$

$\forall g \in G$: $g(Px) = gw = P(gw) = Pg(w + w^\perp) = Pgx$ $\quad P(gw^\perp) = 0$

$gP = Pg$ $\quad$ P also commutes with $\forall g \in G$.

Define $e' = Pe$. then the invariant subspace is defined as

$$W = \{ x e' : x \in R_G \} =: R_G \cdot e'$$

$\left(\forall x \in V: \quad Px = \sum \chi(g)\cdot P(g) = \underbrace{\sum \chi(g)\, g}_{x}(Pe) = xe' \right)$

$e'^2 = e'$ $\quad e'^2 = (Pe)e' = P(ee') = Pe' = e'$

(e' is an idempotent)

Both C_i and P commutes with $\forall g \in G$. is it possible to find P's onto irreps using C_i?

— 8.12.1 Construction of character tables.

Some reasoning for the strategy:

Recall. $[H, T(G)] = 0$

share same set of eigenvectors $\{\varphi_\mu\}$

$$H\varphi_\mu = E_\mu \varphi_\mu$$

$$H\,T(g)\varphi_\mu = T(g) H \varphi_\mu = E_\mu (T(g)\varphi_\mu) \quad \forall g \in G.$$

$W^\mu = \text{span}\{T(g)\varphi_\mu, \forall g \in G\}$ spans an invariant subspace

$$V \cong \bigoplus_\mu W^\mu.$$

We can achieve decomposition of V into smaller invariant subspaces W^μ's, (not necessarily irreps) using different φ_μ's

if W^μ is still reducible. find another operator $[O, T(g)] = 0$. then can further split W^μ.

With a complete set of commuting operators (CSCO). we can find all irreps.

Ref. ① 陈金全 群表示论的新途径

Group representation theory for physicists

② RMP. 57, 211 (1985)

To illustrate the idea, consider a finite group G with r conj. classes $\underline{[C_i]}$, $\underline{|C_i| = m_i}$.

There are also r irreps $\underline{V^\mu}$, with character $\underline{\chi^\mu}$.

Consider the class operators

$$C_i = \sum_{g \in C_i} g$$

① $\forall h \in G$, $[C_i, h] = 0$ (C_i span the center of R_G: $Z(R_G)$)

② $\forall i, j$ $[C_i, C_j] = 0$ because of ①

③ $\underline{\underline{C_i C_j = \sum_{k=1}^{r} C_{ij}^k C_k}}$ ($C_{ij}^k = C_{ji}^k \in \mathbb{N}$)

where C_{ij}^k is the class multiplication coefficient.

Rewrite ③ as

$$\boxed{\hat{C}_i \delta_{C_j} = \sum_{k=1}^{r} [C^i]_{jk} \delta_{C_k}}$$

View C_i as an operator and C_j as a vector. Then we are dealing with an eigen problem.

$[C^i]_{jk}$ is the matrix element of $\hat{C}_i$ in the basis of $\{\delta_{C_j}\}$. (an orthogonal basis set)

$$\langle \delta_{C_j}, \delta_{C_k} \rangle = \frac{1}{|G|} \sum_g \delta_{C_j}(g), \delta_{C_k}(g) = \frac{m_j}{|G|} \delta_{jk}.$$

What is the significance of the eigenvectors of $\hat{C}_i$?

Suppose we find the eigen vectors of $\hat{C}_i$ as $\{\phi^\mu\}$

$$\hat{C}_i \phi^\mu = \lambda_i^\mu \phi^\mu$$

Consider ϕ^ν s.t. $\hat{C}_i \phi^\nu = \lambda_i^\nu \phi^\nu$. then

$$\hat{C}_i(\phi^\mu \phi^\nu) = \lambda_i^\mu(\phi^\mu \phi^\nu) \overset{[C_i, g]=0}{=} \phi^\mu(\hat{C}_i \phi^\nu) = \lambda_i^\nu \phi^\mu \phi^\nu$$

$\Rightarrow$ either $\lambda_i^\mu = \lambda_i^\nu$. or $\phi^\mu \phi^\nu = 0$

~~$\phi^\mu \phi^\nu$ is also an eigenvector with eigenvalue λ_i^μ~~

Assuming λ_i^μ is nondegenerate $\phi^\mu\phi^\nu$ and ϕ^μ are linearly dependent.

$$\phi^\mu \phi^\nu = \alpha_\mu \delta_{\mu\nu} \phi^\mu \qquad (\alpha_\mu \in \mathbb{C})$$

Define $P^\mu = \alpha_\mu^{-1} \phi^\mu$. $\qquad P^\mu P^\nu = \delta_{\mu\nu} P^\mu$

then $C_i = \sum_{\mu=1}^{r} \lambda_i^\mu P^\mu$ is a linear combination of primitive projectors. (i.e. projectors onto different primitive idempotents /ireps)

If there is degeneracy. find another C_i that splits the degeneracy. Continue until all primitive projectors are found.

Now restrict C_i to a specific irrep.

$$C_i^\mu = \lambda_i^\mu \cdot \mathbb{1}_{V^\mu} \qquad \text{(Schur's lemma)}$$

$$\chi_\mu(C_i) = \sum_{g \in C_i} \chi_\mu(g) = m_i \chi_\mu([C_i]) \equiv \mathrm{Tr}_{V^\mu}(C_i) = n_\mu \lambda_i^\mu$$

$$(*) \quad \lambda_i^\mu = \frac{m_i}{n_\mu} \chi_\mu([C_i]) \qquad n_\mu = \dim V^\mu$$

two unknowns: n_μ, χ_μ. need to find another relation:

$$\frac{1}{|G|}\sum_{C_i} m_i\, \chi_\mu(C_i)\, \overline{\chi_\nu(C_i)} = \delta_{\mu\nu}$$

(*)
$$\underbrace{\frac{1}{|G|}\sum_{C_i} m_i\, \lambda_i^\mu\, \overline{\lambda_i^\nu}}_{=:\langle \lambda_i^\nu,\, \lambda_i^\mu \rangle} = \delta_{\mu\nu}\left(\frac{m_i}{n_\mu}\right)^2$$

$$\begin{cases} n_\mu = \dfrac{m_i}{\sqrt{\langle \lambda_i^\mu, \lambda_i^\mu \rangle}} \\[2ex] \chi_\mu = \dfrac{\lambda_i^\mu}{\sqrt{\langle \lambda_i^\mu, \lambda_i^\mu \rangle}} \end{cases}$$

After solving for λ_i^μ's, we obtain both n_μ and χ_μ.