Recap.

$\{ M^{\mu}_{ij} \}$ matrix element of $\overset{\text{unitary}}{\vee}$ irrep V^{μ}

w.r.t. <u>orthogonal basis.</u>

$$\langle M^{\mu}_{ij}, M^{\nu}_{kl} \rangle \equiv \int dg\, \overline{M^{\mu}_{ij}}(g)\, M^{\nu}_{kl}(g)$$

$$= \frac{1}{n_{\mu}} \delta^{\mu\nu} \delta_{ik} \delta_{jl}.$$

$$L^2(G) \cong \bigoplus_{\mu} \mathrm{End}(V^{\mu}) \qquad \text{(Peter-Weyl)}$$

finite $\longrightarrow$ $|G| = \sum_{\mu} n_{\mu}^2$

$$|S_3| = 6 = 1^2 + 1^2 + 2^2$$

$$\Rightarrow \int dg\, \overline{\chi^{\mu}}(g)\, \chi^{\nu}(g) = \delta^{\mu\nu}$$

$\{ \chi^{\mu} \}$ O.N. $\subset L^2(G)$ class

+ completeness

$$V \cong \bigoplus a_{\mu} V^{\mu} \qquad \chi_V = \sum_{\mu} a_{\mu} \chi_{\mu}$$

$$\langle \chi_{\mu}, \chi_V \rangle = a_{\mu}$$

$$\langle \chi_V, \chi_V \rangle = 1 \iff V \text{ irrep}$$

$$\langle \chi_V, \chi_V \rangle = \sum_{\mu\nu} a_{\mu} a_{\nu} \cancel{\langle \chi_{\mu}, \chi_{\nu} \rangle}\ \delta_{\mu\nu}$$

$$= \sum_{\mu} a_{\mu}^2 = 1 \qquad \frac{a_{\mu_0} = 1}{\mu \neq \mu_0 \ a_{\mu} = 0}$$

dim of class function $= \# \{C_i\}$ — $\delta_{C_i} = \begin{cases} 1 & g \in C_i \\ 0 & \end{cases}$

$= \#$ irreps.

$$\frac{1}{|G|} \sum_{\{C_i\}} m_i \overline{\chi_\mu(C_i)}\, \chi_\nu(C_i) = \delta_{\mu\nu}$$

$$k = \# \{C_i\} = \# \text{irreps} \quad \sum_{\{C_i\}} \left(\sqrt{\frac{m_i}{|G|}}\, \overline{\chi_\mu(C_i)} \right) \left(\sqrt{\frac{m_i}{|G|}}\, \chi_\nu(C_i) \right) = \delta_{\mu\nu}$$

$\Rightarrow$ dual orthogonal relation

$$\sum_\mu \overline{\chi_\mu(C_i)}\, \chi_\mu(C_j) = \frac{|G|}{m_i} \delta_{ij}$$

– 8.11.2 character table of finite groups (cont.)

1. $S_2 \cong \mathbb{Z}_2$

	1	$[(12)]$
1^+	1	1
1^-	1	-1

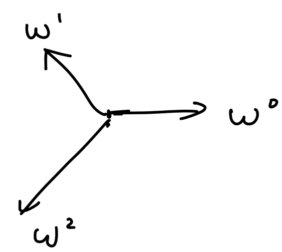
2. $G = \mathbb{Z}_n$ $\quad \#\{C_j\} = n \quad \hat{\mathbb{Z}}_n = \mathbb{Z}_n$

$\#\{\text{irreps}\} = n$

$\mathbb{Z}_3$: $\quad \rho_m(\bar{j}) = \underline{(\omega_m)^j} \qquad \omega_m = e^{i\frac{2\pi}{3}m}$

$= (\omega_1)^{mj} \qquad \omega = e^{i\frac{2\pi}{3}}$

	$[\bar{0}]$	$[\bar{1}]$	$[\bar{2}]$
ρ_0	1	1	1
ρ_1	1	ω	ω^2
ρ_2	1	ω^2	$\omega^{2\times 2} = \omega$



3. $G = S_3$

	$[1]$	$3[(12)]$	$2[(123)]$
1^+	1	1	1
1^-	1	-1	1
2	$\underline{2}$	$\underline{0}$	-1

$2\times 1 + (-1)\times 1\times 2 = 0$

recall. rep of S_3 on $\mathbb{R}^3$ $\quad \{e_1, e_2, e_3\}$

$\phi e_i \to e_{\phi(i)}$

$L = \mathrm{span}\{\sum e_i\}$

$$\mathbb{R}^3 \cong \underbrace{L}_{A} \oplus \underline{L^\perp}$$

$$\chi_{\mathbb{R}^3} \quad , \; 3 \quad , \; 1 \, , \quad 0 .$$

$\chi_{\mathbb{R}^3} = \chi_1 + \chi_2$

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \quad \overset{(12)}{\begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix}} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$a_\mu = \langle \chi_\mu . \chi_{\mathbb{R}^3} \rangle$$

4. V a vector space . $\dim V = d$

S_2 acts on $\underline{V \otimes V}$: $\quad \dim V \otimes V = d^2$

$$\sigma : \quad v_i \otimes v_j \longmapsto v_j \otimes v_i$$

$$\chi_{V\otimes V}(1) = d^2$$

$$\chi_{V\otimes V}(\sigma) = d \qquad (i = j)$$

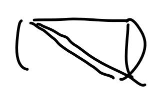
	1	σ
1^+	1	1
1^-	1	-1

$$a_{1^+} = \langle \chi_{1^+} , \chi_{V\otimes V} \rangle = \frac{1}{2}(d + d^2) = \frac{d(d+1)}{2}$$

$$a_{1^-} = \langle \chi_{1^-} , \chi_{V\otimes V} \rangle = \frac{1}{2}(d^2 - d) = \frac{d(d-1)}{2}$$

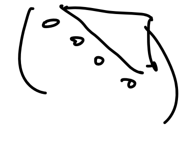
$$V \otimes V = \frac{1}{2} d(d+1) V^{1^+} \oplus \frac{1}{2} d(d-1) V^{1^-}$$

tensors. $T_{ij} \; v_i \otimes v_j$: basis.



symmetric $\quad \frac{1}{2}(e_i \otimes e_j + e_j \otimes e_i)$

antisymmetric $\quad \frac{1}{2}(e_i \otimes e_j - e_j \otimes e_i)$



8.11.3. tensor produces of representations.

V carries space of dim n, basis $\{v_1, \cdots v_n\}$

W m $\{w_1, \cdots w_m\}$

$V\otimes W$. dim $n\cdot m$ basis $\{v_i\otimes w_j \quad 1\le i\le n,\ 1\le j\le m\}$

$$\sum_i a_i v_i \otimes \sum_j b_j w_j = \sum_{ij} a_i b_j \, v_i \otimes w_j$$

G.-action $\quad g\cdot(v\otimes w) := (g\cdot v)\otimes(g\cdot w)$

rep. $\quad (T_1\otimes T_2)(g)(v\otimes w) := T_1(g)\cdot v \otimes T_2(g)\,w.$

mat. rep. $\quad (M_1\otimes M_2)(g)_{ia,jb} = [M_1(g)]_{ij}\,[M_2(g)]_{ab}$

character $\quad \chi_{T_1\otimes T_2} = \chi_{T_1}\cdot\chi_{T_2}$

① particle of spin j_1

j_2 $\Rightarrow V^{j_1}\otimes V^{j_2}$

$$\cong \underline{\bigoplus a_{j_3} V^{j_3}}$$

② many-particle system, local Hilbert space

$\mathcal{H}_i$ spin ½ fermion $=\{\phi, \uparrow, \downarrow, \uparrow\downarrow\}$

$$\mathcal{H} = \bigotimes_i \mathcal{H}_i \Rightarrow \bigoplus_i a_i \mathcal{H}^{\text{config}_i} \quad ——$$

$\uparrow$ N.S.

$\underline{G} \otimes U(1)\otimes SU(2)$

space group

Let (V_1, T_1) and (V_2, T_2) be two representations with isotypic decompositions (over field κ)

$$V_1 = \oplus a_\mu V^\mu \qquad V_2 = \oplus b_\nu V^\nu$$

$$V_1 \otimes V_2 = \oplus a_\mu b_\nu \, \underline{V^\mu \otimes V^\nu}$$

$$V^\mu \otimes V^\nu \cong \bigoplus_\lambda \underline{N_{\mu\nu}^\lambda} V^\lambda$$

$$N_{\mu\nu}^\lambda = \dim_\kappa \mathrm{Hom}_G (V^\lambda, V^\mu \otimes V^\nu)$$

$$\underline{\chi_\mu \cdot \chi_\nu} = \sum_\lambda N_{\mu\nu}^\lambda \chi_\lambda$$

$$N_{\mu\nu}^\lambda = \langle \chi_\lambda, \chi_\mu \cdot \chi_\nu \rangle$$

for Finite groups

$$N_{\mu\nu}^\lambda = \frac{1}{|G|} \sum_{g\in G} \underline{\chi_\mu(g)\chi_\nu(g)} \overline{\chi_\lambda(g)}$$

$m_i = |C_i|$

$$= \frac{1}{|G|} \sum_{\{C_i\}} m_i \chi_\mu(C_i) \chi_\nu(C_i) \overline{\chi_\lambda(C_i)}$$

$$N_{\mu\nu}^\lambda = N_{\nu\mu}^\lambda \qquad (V^\mu \otimes V^\nu \cong V^\nu \otimes V^\mu)$$

Examples. 1. ρ_m of $\mathbb{Z}_N$ $\quad \rho_m(j) = \left(e^{i\frac{2\pi}{N}m}\right)^j$

$$\rho_m \otimes \rho_n \cong \rho_{m+n}$$

$$N_{mn}^\lambda = \frac{1}{N} \sum_{l} \underline{e^{i\frac{2\pi}{N}(m+n)l} e^{-i\frac{2\pi}{N}\cdot\lambda l}}$$

$$= \delta_{m+n,\lambda}$$

2. irreps of S_3.

$$V^{1^+} \otimes V^{\mu} \cong \oplus_{\lambda} N_{1^+,\mu}^{\lambda} V^{\lambda}$$

$$N_{1^+,\mu}^{\lambda} = \frac{1}{|G|} \sum m_i \underline{\chi_{\mu}}(C_i) \overline{\underline{\chi_{\lambda}}(C_i)}$$

$$= \delta_{\mu\lambda}$$

$$\bigoplus_{\lambda} \delta_{\mu,\lambda} V^{\lambda} = V^{\mu}$$

$$\Rightarrow \underline{V^{1^+} \otimes V^{\mu}} \cong V^{\mu}$$

<u>check</u>

$$V^- \otimes V^- \cong V^+$$

$$V^- \otimes V^2 \cong V^2$$

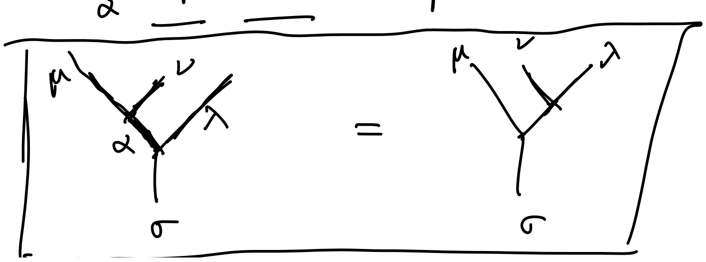
$$V^2 \otimes V^2 \cong V^+ \oplus V^- \oplus V^2$$

$$\underline{(V^{\mu} \otimes V^{\nu}) \otimes V^{\lambda} \cong V^{\mu} \otimes (V^{\nu} \otimes V^{\lambda})}$$

$$\text{LHS} \cong \oplus_{\alpha} \underline{D_{\mu\nu}^{\alpha}} \, \underline{V^{\alpha}} \otimes \underline{V^{\lambda}}$$

$$\cong \oplus_{\sigma} (\oplus_{\alpha} \underline{D_{\mu\nu}^{\alpha} \otimes D_{\alpha\lambda}^{\sigma}}) \otimes V^{\sigma} \cong \oplus_{\sigma} (\oplus_{\beta} D_{\nu\lambda}^{\beta} \otimes D_{\mu\beta}^{\sigma}) \otimes V^{\sigma}$$

$$\sum_{\alpha} \underline{N_{\mu\nu}^{\alpha}} \, \underline{N_{\alpha\lambda}^{\sigma}} = \sum_{\beta} N_{\mu\beta}^{\sigma} N_{\nu\sigma}^{\beta}$$



"F-move"

digression : "$\underline{\text{Category theory}}$"

TQFT / anyons / top. quantum computation

$(X \otimes Y) \otimes (Z \otimes W)$ → $\underline{\text{pentagon relation}}$

(ref. PRB 100. 115147)

8.12 Explicit decomposition of a representation

recall $S_2 \cong \mathbb{Z}_2$ $\quad T(\sigma) v \to -v$

$$P = \frac{1}{2}(1 \pm T)$$

Let (T, V) be any rep. of a compact group G. Define

$$P_{ij}^{(\mu)} := n_\mu \int_G \overline{\mathcal{U}_{ij}^{(\mu)}(g)}\, T(g)\, dg$$

$\mathcal{U}_{ij}^{\mu}$ w.r.t unitary irreps with ON basis of $V^{(\mu)}$

$$\boxed{P_{ij}^{(\mu)} P_{kl}^{(\nu)} = \delta^{\mu\nu}\delta_{jk} P_{il}^{(\nu)}}$$

$$T(h)\, P^{\mu}_{ij} = n_\mu T(h) \int_G dg\, \overline{\mathcal{M}^{(\mu)}_{ij}(g)}\, T(g)$$

$$= n_\mu \int_G dg\, \overline{\mathcal{M}^{(\mu)}_{ij}(g)}\, T(hg)$$

$$\overset{hg \to g}{=} n_\mu \int_G dg\, \underbrace{\overline{\mathcal{M}^{(\mu)}_{ij}(h^{-1}g)}}_{\mathcal{M}^{(\mu)}_{ki}(h)\, \overline{\mathcal{M}^{(\mu)}_{kj}(g)}}\, T(g)$$

$$= \sum_k^{n_\mu} \mathcal{M}^{\mu}_{ki}(h)\, P^{(\mu)}_{kj} \cancel{(g)}$$

$$T(h)\, P^{\mu j}_{i} = \sum_k \mathcal{M}^{\mu}_{ki}(h)\, P^{\mu j}_{k}$$

$\forall \varphi \in V.$ $(P^{\mu}_{ij}\varphi \neq 0)$. then

span $\{ \underline{P^{\mu}_{ij}\varphi}, \; i=1, \cdots n_\mu \}$ (fix $\mu . j$)

transforms as (T^{μ}, V^{μ})

$$P_\mu := \sum_{i=1}^{n_\mu} P^{(\mu)}_{ii} = \underline{n \int_G dg\, \overline{\chi_\mu(g)}\, T(g)}$$

$$P_\mu P_\nu = \sum_{i=1}^{n_\mu} \sum_{j=1}^{n_\nu} P^{\mu}_{ii} \underline{P^{\nu}_{jj}} = \delta_{\mu\nu} \sum_{ij} \delta_{ij} P^{\nu}_{ij} = \delta_{\mu\nu} P_\nu$$

<u>Example</u>. 1 $\quad P = \underline{\int_G T(g)\, dg}$ $\quad$ trivial rep.

$$T(h) P = \int_G T(h) T(g)\, dg = P$$

$$\underline{T(h)}(P\varphi) = (P\varphi) \quad \forall \varphi.$$

⑧

2. HW. S_3 $\quad \mathbb{R}^3 \cong V_{1+} \oplus V_2$

Character table for point group D_4

D_4	E	$2C_4$ (z)	C_2 (z)	$2C'_2$	$2C''_2$	linear functions, rotations	quadratic functions	cubic functions
A_1	+1	+1	+1	+1	+1	-	x^2+y^2, z^2	-
A_2	+1	+1	+1	-1	-1	z, R_z	-	$z^3, z(x^2+y^2)$
B_1	+1	-1	+1	+1	-1	-	x^2-y^2	xyz
B_2	+1	-1	+1	-1	+1	-	xy	$z(x^2-y^2)$
E	+2	0	-2	0	0	(x, y) (R_x, R_y)	(xz, yz)	(xz^2, yz^2) (xy^2, x^2y) (x^3, y^3)

$$\varphi(x, y, z) \xrightarrow{\sigma_x} \varphi(-x, y, z)$$

$$\varphi(x, y, z) = \alpha x + \beta y + \gamma z \xrightarrow{P_{ij}^{A^2}} \alpha = \beta = 0$$