HW.

(a) $\alpha = e^{i\frac{1}{2}(\phi+\psi)} \cos\theta/2 \qquad \beta = i e^{i\frac{1}{2}(\phi-\psi)} \sin\theta/2$

$$g = \begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix} \in SU(2) \qquad \begin{array}{l} \phi \in [0, 2\pi) \\ \theta \in [0, \pi) \\ \psi \in [0, 4\pi) \end{array}$$

$$\alpha = r e^{\frac{i}{2}(\phi+\psi)} \cos\frac{\theta}{2} \Big|_{r=1}$$

$$d\alpha\, d\bar\alpha\, d\beta\, d\bar\beta = \left| \frac{\partial(\alpha, \bar\alpha, \beta, \bar\beta)}{\partial(r, \psi, \phi, \theta)} \right|_{r=1} d\psi\, d\phi\, d\theta$$

$$= \underline{\left(\tfrac{1}{2} r^3 \sin\theta\right)}\Big|_{r=1}\, d \text{———}$$

$g \to \underline{g_0 g} \qquad \underline{|\det g_0| = 1}$

$$C \int \underbrace{\sin\theta\, d\theta}_{2}\, \underbrace{d\phi}_{2\pi}\, \underbrace{d\psi}_{4\pi} = 1 \qquad C = \frac{1}{16\pi^2}$$

(b) $\underline{\int_{SU(2)} dg\, g_{\alpha\beta} = 0} \qquad \underline{\int_{SU(2)} dg\, g_{\alpha\beta}\, g_{\gamma\delta} = \tfrac{1}{2}\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}.}$

$$\underline{\phi_{\alpha\beta}} \overset{\text{L.I.}}{=} \int dg\, (g_0 g)_{\alpha\beta} = \underline{(g_0)_{\alpha\gamma}} \underbrace{\int dg\, g_{\gamma\beta}}_{\phi_{\gamma\beta}}$$

fix β.

$$g_0 \cdot \begin{pmatrix} \phi_{0\beta} \\ \phi_{1\beta} \end{pmatrix} = \begin{pmatrix} \phi_{0\beta} \\ \phi_{1\beta} \end{pmatrix}$$

$$g_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SU(2) \qquad \phi_{0\beta} = \pm \phi_{1\beta} = 0$$

$$(A^{\beta\delta})_{\alpha\gamma} = \int dg\, g_{\alpha\beta}\, g_{\gamma\delta} = \int dg\, (g_0 g)_{\alpha\beta} (g_0 g)_{\gamma\delta} .$$

$$= (g_0)_{\alpha s} \underbrace{\int dg\, g_{s\beta}\, g_{t\delta}}_{(A^{\beta\delta})_{st}} (g_0)_{\gamma t}$$

$$\Rightarrow \quad A^{\beta\delta} = g_0 \cdot A^{\beta\delta} \cdot g_0^T \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$g_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \Rightarrow \quad a = d . \quad b = -c$$

$$g_0 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \Rightarrow \quad a = -d \quad b = -c$$

$$A^{\beta\delta} = c^{\beta\delta} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \to \epsilon$$

$$A^{\alpha\gamma} = c^{\alpha\gamma} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$A_{\alpha\gamma,\beta\delta} = c_{\beta\delta}\, \epsilon_{\alpha\gamma} = c_{\alpha\gamma} \cdot \epsilon_{\beta\delta}$$

$$A = c \cdot \epsilon_{\alpha\gamma}\, \epsilon_{\beta\delta}$$

$$I = \int dg\, g \cdots g \xrightarrow{\text{L. tran.}} \int dg\, (g_0 g)(g_0 g) \cdots (g_0 g)$$

$$g_0 = -\mathbb{1}_2 \qquad I = (-1)^n I \quad \Rightarrow \quad n \text{ even}$$

$$g_0 = \begin{pmatrix} e^{-i\theta} & \\ & e^{i\theta} \end{pmatrix} \quad \Rightarrow \quad I = e^{i\theta \Sigma (-1)^{\alpha_i}} I$$

$$\Rightarrow \Sigma (-1)^{\alpha_i} = 0$$

half $\alpha_i = 1$. half $= 2$

Recap. Peter – Weyl.

① irreps of compact / finit groups are finite dimensional.

② matrix elements of unitary irreps are orthogonal basis of $L^2(G)$

$$\boxed{\langle M^{\mu}_{ij}, M^{\nu}_{i'j'} \rangle = \frac{1}{n_\mu} \delta_{\mu\nu} \delta_{ii'} \delta_{jj'}}$$

$$\langle \varphi_1, \varphi_2 \rangle := \int_G \overline{\varphi_1(g)}\, \varphi_2(g)\, dg$$

$$\boxed{\bigoplus_\mu \operatorname{End} V^{(\mu)} \longrightarrow L^2(G)}$$

$$\oplus S \longmapsto \sum \operatorname{Tr}_V (S T(g^{-1}))$$

$$\oplus e_{ij} \longmapsto \sum M^{\mathrm{Tr}\cdot -1}_{ij}$$

$$\tilde{A} = \int T^{\nu}(g)\, A\, T^{\mu}(g^{-1})\, dg$$

less abstract refs:

- Zee. GT in a nutshell. Chap II.2
- Dresselhaus, GT. Sec. 2.7

in Dresselhaus $\quad M = \sum_g D^{(\nu)}(g)\, X\, D^{(\mu)}(g^{-1}), \quad X$ matrix

$$D^{\nu}(g) M = M D^{\mu}(g) \quad \forall g \in G.$$

More abstract.

Sepanski "Compact Lie Groups". Springer (GTM 235)

Chap 3. "Harmonic Analysis"

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①

$$\bigoplus_\mu \mathrm{End}\, V^\mu \cong L^2(G)$$

↪ Corollary for finite groups.

$L^2(G)$ of dim $|G|$:

$$\delta_a(g) = \begin{cases} 1 & g = a \\ 0 & \end{cases}$$

$$(g\delta_a = \delta_{ga})$$

$$\forall f : G \to \mathbb{C}. \qquad f = \sum_{g_0 \in G} f(g_0)\delta_{g_0}$$

$$\mathrm{End}(V^\mu) \cong \mathrm{Mat}_{n_\mu \times n_\mu}(\mathbb{C}) \qquad e_{ij}$$

$$\dim_{\mathbb{C}}(\mathrm{End}(V^\mu)) = n_\mu^2$$

$$\Rightarrow |G| = \sum_\mu n_\mu^2$$

Examples 1. S_3 $\qquad |S_3| = 6 \qquad \rho_1, \rho_1', \rho_2$

$$6 = 1^2 + 1^2 + 2^2$$

$$S_3^{\text{reg.rep}} \cong \rho_1 \oplus \rho_1' \oplus 2\rho_2$$

$\times$ $6 = 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2$ ($\because S_3$ nonabelian)

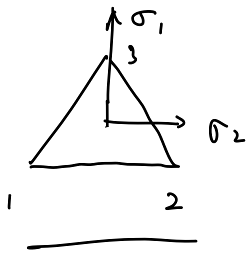
2

$\underline{\underline{S_3}}$

1. $M^+(\phi) = 1 \qquad \forall \phi \in S_3$

2. $M^-(\phi) = 1 \qquad \phi \in \{(\,),\ (123),\ (132)\} = A_3$

   $M^-(\phi) = -1 \qquad \phi \in \{(12),\ (13),\ (23)\}$

3. 

$$\underline{M^{(2)}(12)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$M^{(2)}(13) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$$M^{(2)}((123)) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

$$\langle M^{\mu}_{ij}, M^{\mu'}_{i'j'} \rangle = \frac{1}{n_\mu} \delta_{\mu\mu'} \delta_{ii'} \delta_{jj'}$$

a. $\langle M^+, M^- \rangle = 0$

b. $\langle M^+, M^{(2)}_{11} \rangle = \frac{1}{6} \sum M^{(2)}_{11}(g) = \frac{2}{6}(1 - \frac{1}{2} - \frac{1}{2}) = 0$

c. $\langle M^{(2)}_{11}, M^{(2)}_{11} \rangle = \frac{2}{6}(1 + \frac{1}{4} + \frac{1}{4}) = \frac{1}{2} = \frac{1}{n_\mu}$

2. $G = \mathbb{Z}_2 = \langle \sigma \mid \sigma^2 = 1 \rangle$

$\varphi \in L^2(G) = \{ Map(G, \mathbb{C}) \}$

$\varphi(1) = \varphi_+ \in \mathbb{C}$

$\varphi(\sigma) = \varphi_-$

$L^2(G) \cong \mathbb{C}^2$

$\mathbb{Z}_2$ irreps $\rho_\pm(\sigma) = \pm 1 \qquad V_\pm \cong \mathbb{C}$

$$\begin{cases} M^+(1) = M^-(1) = 1 \\ M^+(\sigma) = 1 \quad M^-(\sigma) = -1 \end{cases}$$

$$\varphi = \frac{\varphi_+ + \varphi_-}{2} M^+ + \frac{\varphi_+ - \varphi_-}{2} M^-$$

$$\Rightarrow \begin{cases} \varphi(1) = \varphi_+ \\ \varphi(\sigma) = \varphi_- \end{cases}$$

Previously : $T(\sigma)$

$P_\pm = \frac{1}{2}(1 \pm T(\sigma))$ is of the

form : $P_\pm = \int_G \overline{\mu^\pm(g)}\, T(g)\, dg$ (later)

3. $G = U(1)$ $\quad (\hat{G} = \mathbb{Z})$

(ρ_n, V_n) : $\rho_n(z) = z^n \quad n \in \mathbb{Z}. \quad (= e^{i\theta n}$ $)$ $\quad \theta \in [0, 2\pi)$

$V_n \cong \mathbb{C}$

$$\langle \varphi_1, \varphi_2 \rangle = \int_0^{2\pi} \frac{d\theta}{2\pi} (\varphi_1(\theta))^* \varphi_2(\theta)$$

$$e^{i\theta(n_1 - n_2)}$$

$\{e^{i\theta n}\}$ ON basis

$$\Psi = \sum_n \hat{\Psi}_n \rho_n \qquad \hat{\Psi} = \int_{U(1)} \rho_n^* \Psi(g)\, dg$$

8.11. Orthogonality relations of characters ; Charater table.

8.11.1 Orthogonality relations —

Recall. a class function on G:

$$f : G \to \mathbb{C}.$$

$f(g) = f(hgh^{-1})$ $\forall g, h \in G$. They span a subspace $L^2(G)^{class} \subset L^2(G)$.

<u>Theorem</u> The characters $\{\chi_\mu\}$ is an orthonormal (ON) basis for the vector space of class functions $L^2(G)^{class}$.

<u>Proof</u>. $\int_G dg\, M^{(\mu)}_{ij}(g)^* M^{(\nu)}_{kl}(g) = \frac{1}{n_\mu}\delta_{\mu\nu}\delta_{ik}\underline{\delta_{jl}}$

set $i=j$. $k=l$ & sum over i,k

$$\Rightarrow \int_G dg\, M^{(\mu)}_{ii}(g)^* M^{(\nu)}_{kk}(g) = \underline{\frac{1}{n_\mu}}\delta_{\mu\nu}\underline{\delta_{ik}}$$

$$\xRightarrow{\sum_{i,k}} \int_G dg\, \chi^\mu(g)^* \chi^\nu(g) = \delta_{\mu\nu}$$

$\Rightarrow \{\chi_\mu\}$ ON set

<u>Completeness ?</u>

$$\forall f \in L^2(G) \xrightarrow[\{M^\mu_{ij}\} \text{ complete}]{\text{Peter-Weyl}} f(g) = \sum_{\mu,i,j} \hat{f}^\mu_{ij} M^\mu_{ij}(g)$$

of $f \in L^2(G)^{class}$. $f(g) = f(hgh^{-1})$

$$\underbrace{\int_G dh\, f(g)}_{f(g)} = \int_G dh\, f(hgh^{-1})$$

⑤

$$\int_G f(hgh^{-1})\,dh = \sum_{\mu,i,j} \hat{f}^{\mu}_{ij} \int_G \underbrace{M^{\mu}_{ij}(hgh^{-1})}_{M^{\mu}_{ik}(h) M^{\mu}_{kl}(g) M^{\mu}_{lj}(h^{-1})}\,dh$$

$$= \sum_{\substack{\mu,i,j\\ k,l}} \hat{f}^{\mu}_{ij} M^{\mu}_{kl}(g) \underbrace{\int_G M^{\mu}_{ik}(h) M^{\mu\,*}_{jl}(h)\,dh}_{\frac{1}{n_\mu}\delta_{ij}\delta_{kl}}$$

$$= \sum_{\mu,i} \frac{\hat{f}^{\mu}_{ii}}{n_\mu} \chi_\mu(g)$$

$$\Rightarrow f(g) = \sum_{\mu,i} \frac{\hat{f}^{\mu}_{ii}}{n_\mu} \chi_\mu(g)$$

$\Rightarrow$ $\{\chi_\mu\}$ spans full $L^2(G)^{\text{class}}$.

isotypic decomposition of some rep V.

$$V \cong \oplus a_\mu V^{(\mu)}$$

$$\boxed{\begin{aligned} &\Rightarrow \chi_V = \sum_\mu a_\mu \chi_\mu \\ &a_\mu = \langle \chi_\mu, \chi_V \rangle = \int_G \overline{\chi_\mu(g)}\, \chi_V(g)\, dg \end{aligned}}$$

if $V \cong L^2(G)$ of a finite group.

$$\chi_V(e) = \dim V = |G|$$

$$\chi_V(g \neq e) = 0$$

$$a_\mu = \frac{1}{|G|} \sum_g \overline{\chi_\mu(g)}\, \chi_V(g) = \frac{1}{|G|} \cdot n_\mu \cdot |G| = n_\mu$$

$$|G| = \sum_\mu a_\mu \dim V^\mu = \sum_\mu n_\mu \cdot n_\mu = \sum_\mu n_\mu^2$$

8.11.2. Character table (finite groups)

	( )	(12)	(123)
	Conj	classes	
Γ^1			
Γ^2			
$\vdots$			

For finite groups,

we can define a set of class functions

$$\delta_{C_i}(g) = \begin{cases} 1 & g \in C_i \\ 0 & \end{cases}$$

where C_i is a distinct conjugacy class.

$\{\delta_{C_i}\}$ is also a basis for the class functions $L^2(G)^{\text{class}}$

From above, $\{\chi_\mu\}$ is a basis of $L^2(G)^{\text{class}}$

Theorem. The number of conjugacy classes of a finite group G = the number of irreps.

The character table is an $r \times r$ matrix

$m_r = |C_r|$

	$m_1 C_1$ (E)	$m_2 C_2$	$\cdots$	$m_r C_r$
trivial Γ^1 V^1	$\chi_1(C_1)$	$\chi_1(C_2)$	$\cdots$	
irreps $\Rightarrow$ V^2	$\chi_2(C_1)$	$\chi_2(C_2)$	$\cdots$	
$\vdots$	$\vdots$	$\vdots$		$\vdots$
V^r	$\vdots$	$\vdots$		$\chi_r(C_r)$

$$\int_G dg\, \overline{\chi_\mu(g)}\, \chi_\nu(g) = \delta_{\mu\nu} \quad \Rightarrow$$

$$\frac{1}{|G|}\sum_{\{C_i\}} m_i \overline{\chi_\mu(C_i)}\, \chi_\nu(C_i) = \delta_{\mu\nu}$$

define $S_{\mu i} = \sqrt{\frac{m_i}{|G|}}\,\chi_\mu(C_i)$ then

$$\sum_{i=1}^{r} S_{\mu i} S^*_{\nu i} = \delta_{\mu\nu}.$$

S is a unitary matrix

$$\underline{S \cdot S^\dagger = \mathbb{1}_r}$$

There is a dual orthogonality relation

$$\sum_{\mu} \overline{\chi_\mu(C_i)}\, \chi_\mu(C_j) = \frac{|G|}{m_i}\delta_{ij}$$

<u>Examples</u> 1. $S_2 \cong \mathbb{Z}_2$

	[1]	[(12)]
1^+	1	1
1^-	1	-1