Recap on Pontryagin duals

1. Let S be an abelian group.

① $\hat{S} := \mathrm{Hom}(S, U(1))$ the character group of S

$$\chi : \begin{aligned} S &\longrightarrow U(1) \\ s &\longmapsto \chi(s) \end{aligned}$$

$$(\chi_1 \cdot \chi_2)(s) = \chi_1(s)\,\chi_2(s)$$

a. $S = \mathbb{R}$ $\qquad \chi_k(a) = e^{ika} \qquad a \in \mathbb{R}$

$$\chi_{k_1} \cdot \chi_{k_2} = \chi_{\underline{k_1+k_2}}$$

$$\Rightarrow \hat{\mathbb{R}} \cong \mathbb{R}$$

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b. $S = \mathbb{Z}_n$ $\qquad \chi(T) = \omega \qquad \omega^n = 1 \qquad \omega_k = e^{i\frac{2\pi}{n}k}$

$k \in [1, n]$

$$\chi_{k_1} \cdot \chi_{k_2} = \chi_{k_1+k_2}$$

$$\Rightarrow \hat{\mathbb{Z}}_n \cong \mathbb{Z}_n$$

c. $S = \mathbb{Z}$. $\qquad \chi_k(1) = e^{ik} \qquad \chi_k(n) = e^{ikn} = e^{i(k+2\pi)n}$

$k \sim k + 2\pi$

$k \in \mathbb{R}/\sim$

$$\chi_{k_1} \cdot \chi_{k_2} = \chi_{k_1+k_2}$$

$$\Rightarrow \hat{\mathbb{Z}} \cong U(1)$$

d. $S = U(1)$ $\qquad \omega = e^{i\phi} \qquad \phi \in [0, 2\pi)$

$$\chi_k(\phi + 2\pi \cdot n) = e^{ik(\phi + 2n\pi)}$$

$$\Rightarrow k \in \mathbb{Z}$$

$$\Rightarrow \hat{U(1)} = \mathbb{Z}$$

② $\hat{\hat{S}} := \mathrm{Hom}(\hat{S}, U(1)) = \mathrm{Hom}(\mathrm{Hom}(S, U(1)), U(1))$

$$\begin{aligned} \hat{S} &\longrightarrow U(1) \\ ev_s : \chi &\longmapsto \chi(s) \in U(1) \end{aligned}$$

③ Locally compact S

$$\hat{\hat{S}} \cong S$$

$$\begin{aligned} S &\longrightarrow \hat{\hat{S}} \\ s &\longmapsto ev_s \end{aligned}$$

$$\left\{ \begin{array}{ll} \hat{\hat{\mathbb{R}}} = \hat{\mathbb{R}} = \mathbb{R} & \\ \hat{\hat{\mathbb{Z}}}_n = \mathbb{Z}_n & \\ \hat{\mathbb{Z}} = U(1) & \hat{\hat{\mathbb{Z}}} = \mathbb{Z} \\ \widehat{U(1)} = \mathbb{Z} & \hat{\hat{U}}(1) = U(1) \end{array} \right.$$

(non-locally compact . $\hat{\mathbb{Q}} = \mathbb{R}$ $\quad \hat{\hat{\mathbb{Q}}} = \mathbb{R} \neq \mathbb{Q}$)

2 Fourier transform.

$$\hat{f}(\chi) = \int_G dg\, f(x)\, \chi(x)$$

$$f(x) = \int_{\hat{G}} d\hat{g}\, \hat{f}(\chi)\, \overline{\chi(x)}$$

$\left(\dfrac{\overline{\chi(x)}}{\chi(x)} \quad \text{matter of convention} \right)$

3. $\hat{\mathbb{Z}} \cong U(1)$

$\Gamma \cong \mathbb{Z}^d \qquad T = \mathbb{R}^d/\Gamma \cong U(1)^d$ torus / unit cell

$\Gamma^\vee = \{ g \in \mathbb{R}^d . \ g\gamma \in \mathbb{Z} \} \cong \mathbb{Z}^d$ reciprocal lattice

$T^\vee = \mathbb{R}^d / \Gamma^\vee \cong U(1)^d$ Brillouin zone torus

$\cong \widehat{\Gamma}$

$\bar{k} \in T^\vee \cong \widehat{\Gamma}$ $\bar{k}$ labels different irreps of the discrete translation group (Γ)

$\chi_{\bar{k}}(\gamma) = e^{2\pi i \vec{k}\cdot\vec{\gamma}}$ ($\vec{\gamma} \in \Gamma$

$\vec{k} = \bar{k} + \vec{g}$ $\vec{g} \in \Gamma^\vee$)

8.9. Pontryagin duality

8.9.1. Application: Bloch's theorem

(see also. e.g. Ashcroft & Mermin
Solid state physics, Chap. 8)

The one-electron Hamiltonian.

$$H = -\frac{\hbar^2}{2m}\nabla^2 + U(r) \qquad U(r) = U(r+\gamma)$$

$$(\gamma \in \Gamma)$$

Bloch's theorem: The eigen states φ of the above Hamiltonian H can be chosen to have the form

$$\varphi(\vec{r}) = e^{i\vec{k}\cdot\vec{r}}\, u(\vec{r})$$

with $u(r+\gamma) = u(r)$

Define the translation operator $T(\gamma)$

$$T(\gamma)\varphi(x) = \varphi(x+\gamma)$$

The eigen states of H are 1D irreps of the translation group

$$T(\gamma)\varphi(x) = \chi_k(\gamma)\varphi(x) \equiv \varphi(x+\gamma)$$

write $\underline{\varphi(x) = e^{2\pi i \vec{k}\cdot\vec{x}}\, u_k(x)}$

$$e^{2\pi i k(x+\gamma)}\, u_k(x+\gamma) = \underline{e^{2\pi i k\cdot\gamma}} \cdot e^{2\pi i k\cdot x}\, u_k(x)$$

Bloch's theorem: $\Rightarrow \underline{u_k(x+\gamma) = u_k(x) \qquad (\forall \gamma \in \Gamma)}$

The Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d)$ is isotypicaly decomposed as

$$\mathcal{H} \cong \oint_{T^\vee} d\bar{k}\, \mathcal{H}_{\bar{k}}$$

$$\mathcal{H}_{\bar{k}} \text{ spanned by } \{\varphi(x) : \underset{\underset{T(\gamma)\cdot\varphi(x)}{\equiv}}{\varphi(x+\gamma)} = \chi_{\bar{k}}(\gamma)\varphi(x)\}$$

The eigenvalue problem

$$H\varphi(x) = E\varphi(x) \qquad \underline{x \in \mathbb{R}^d.}$$

$\forall k \in \bar{k}$
$$\Rightarrow H e^{2\pi i kx} u_k(x) = E_k\, \underline{e}^{2\pi i kx} u_k(x)$$

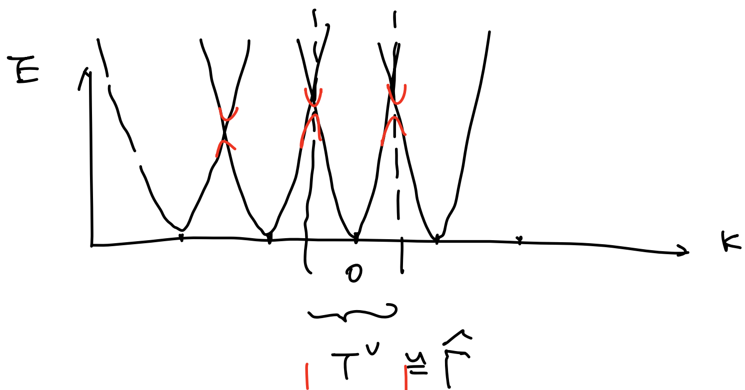
$$H_k u_k(x) = E_k u_k(x)$$

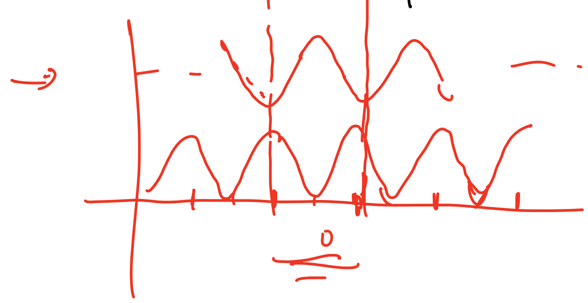
$$\text{with } H_k = e^{-2\pi i kx} H e^{2\pi i kx}$$

H_k acts on L^2-functions on $T = \mathbb{R}^d/\Gamma$

$$H_k = U H_{k'} U^{-1} \qquad \begin{cases} U = e^{2\pi i g x} \\ k' = k + g \end{cases}$$

spectrum over different $\bar{k}$ is the bandstructure.





§10. Orthogonality relations of matrix elements of reps ; Peter-Weyl theorem.

Recall : ① Basics of reg. rep.

$$L^2(G) = \{ f : G \to \mathbb{C} \mid \int_G |f(g)|^2 dg < \infty \}$$

is a unitary $G \times G$

② V a rep. $\mathrm{End}(V) := \mathrm{Hom}(V, V)$ is also a unitary rep of $G \times G$.

$S \in \mathrm{End}(V)$: $(g_1, g_2) \cdot S = T(g_1) \cdot S \cdot T(g_2)^{-1}$

$$\iota: \mathrm{End}(V) \longrightarrow L^2(G)$$

$$S \longmapsto \underline{\mathrm{Tr}_V(S\,T(g^{-1}))} := \varphi_S$$

matrix unit $\underline{e_{ij}} \longmapsto M^{\mathrm{Tr},-1}_{ij} = M(g^{-1})_{ji}$

$$\iota: \underline{\bigoplus_\mu \mathrm{End}(V^\mu)} \longrightarrow \underline{L^2(G)}$$

$$\bigoplus_i S_i \longmapsto \sum_i \underline{\varphi_{S_i}}$$

<u>Peter-Weyl theorem</u>: G <u>compact</u>. Then there is an isomorphism of $G\times G$ representations

$$L^2(G) \cong \bigoplus_\mu \mathrm{End}(V^{(\mu)})$$

where we sum over the distinct isomorphism class of each irrep <u>exactly once</u>.

Peter-Weyl theorem is the consequence of two statements.

1. Let (V, T) be a <u>unitary irrep</u> of a compact group G on a complex vector space V.

   Then V is finite dimensional.

   (for a proof see GM notes)

2. Let G be a compact group. The Hermition inner product on $L^2(G)$

$$\langle \varphi_1 , \varphi_2 \rangle := \underline{\int_G \varphi_1^*(g) \varphi_2(g) \, dg}$$

with normalized Haar measure. s.t. the volume of G $\int_G dg = 1$.

$\underline{L^2(G) \cong \oplus a^\mu V^{(\mu)}}$ Let $\{V^\mu\}$ be a set of representations of $\underline{\text{distinct}}$ isomorphism classes of unitary irreps.

(Because of statement 1). For each $V^{(\mu)}$ choose an orthonormal (ON) basis $w_i^{(\mu)}$.

$i = 1, \cdots, n_\mu$. $n_\mu = \dim V^{(\mu)}$

$$T^{(\mu)}(g) w_i^{(\mu)} = \sum_{j=1}^{n_\mu} M_{ji}^\mu(g) w_j^{(\mu)}$$

M_{ij}^μ form a complete orthogonal set of functions on $L^2(G)$.

$$\langle M^{\mu_1}_{i_1, j_1} , M^{\mu_2}_{i_2, j_2} \rangle = \frac{1}{n_\mu} \delta^{\mu_1 \mu_2} \delta_{i_1, i_2} \delta_{j_1, j_2}$$

$\underline{\text{Proof}}$. $\forall A : V^\mu \to V^\nu$

$$\tilde{A} := \int_G T^\nu(g) A T^\mu(g^{-1}) \, dg$$

$$T^\nu(h) \tilde{A} = \int_G T^\nu(hg) A T^\mu(g^{-1}) \, dg$$

$$\overset{g \to h^{-1}g}{=} \int_G T^{\nu}(g)\, A\, T^{\mu}((h^{-1}g)^{-1})\, dg$$

$$= \left(\int_G T^{\nu}(g)\, A\, T^{\mu}(g)^{-1}\, dg \right) T^{\mu}(h)$$

$$= \tilde{A}\, T^{\mu}(h)$$

$\tilde{A}$ is an intertwiner

$$\begin{tikzcd} V^{\mu} \arrow[r, "\tilde{A}"] \arrow[d, "T^{\mu}"] & V^{\nu} \arrow[d, "T^{\nu}"] \\ V^{\mu} \arrow[r, "\tilde{A}"] & V^{\nu} \end{tikzcd}$$

By Schur's lemma. $\tilde{A} = \delta_{\mu\nu} \hat{A}$. $\underline{\hat{A} = C_A \mathbb{1}_{\nu}}$

Assign a basis for V^{μ} and V^{ν}

$$[\tilde{A}]_{ia} = \underline{\delta_{\mu\nu} C_A \cdot \delta_{ia}} = \int_G dg\, [M^{\nu}(g)\, A\, M^{\mu}(g^{-1})]_{ia}$$

$$= \underline{\sum_{i',a'} \int_G dg\, M^{\nu}_{ii'}(g)\, A_{i'a'}\, M^{\mu}_{a'a}(g^{-1})} \quad (*)$$

set $\mu = \nu$, $i = a$, and take the trace.

$$n C_A = \sum_{i,i',a'} \int_G dg\, M^{\mu}_{ii'}(g)\, A_{i'a'}\, M^{\mu}_{a'i}(g^{-1})$$

$$= \int_G dg\, \mathrm{Tr}\left(\cancel{M^{\mu}(g)}\, A\, \cancel{M^{\mu}(g^{-1})} \right)$$

$$= \int_G dg\, (\mathrm{Tr}\, A) = \mathrm{Tr}\, A$$

$$\Rightarrow \underline{C_A = \frac{1}{n_{\mu}} \mathrm{Tr} A}$$

Now take A to be the matrix unit e_{jk}

($\mathrm{Tr}\, e_{jk} = \delta_{jk}$).

insert into (*)

$$\sum_{i'a'} \int_G dg\, M^{\nu}_{ii'}(g) \overset{[e_{jk}]_{i'a'}}{\overbrace{(\delta_{ji'}\delta_{ka'})}} M^{\mu}_{a'a}(g^{-1}) = \overset{\frac{\delta_{jk}}{n_\mu}}{\overbrace{\frac{\mathrm{Tr}\, e_{jk}}{n_\mu}}} \delta_{\mu\nu}\delta_{ia}$$

$$\Rightarrow \int_G dg\, \underline{M^{\nu}_{ij}(g)}\, \underline{M^{\mu}_{ka}(g^{-1})} = \frac{1}{n_\mu}\delta_{\mu\nu}\delta_{ia}\delta_{jk}$$

$$\Downarrow$$

$$[M^{\mu}(g)^{+}]_{ka} = \overline{M^{\mu}_{ak}(g)}$$

$$\Rightarrow \langle M^{\mu}_{ak}, M^{\nu}_{ij} \rangle = \frac{1}{n_\mu}\delta_{\mu\nu}\delta_{ia}\delta_{jk}$$

$$\Rightarrow \langle M^{\mu_1}_{i_1, j_1}, M^{\mu_2}_{i_2, j_2} \rangle = \frac{1}{n_\mu}\delta^{\mu_1\mu_2}\delta_{i_1 i_2}\delta_{j_1 j_2}$$

We have shown that $\{M^{\mu}_{ij}\}$ is a set of orthogonal functions on $L^2(G)$.

<u>basis $\Leftarrow$ completeness ?</u>

Let $\underline{W}$ be the subspace spanned by $\{M^{\mu}_{ij}\}$.

$\Rightarrow$ The orthogonal complement $\underline{\underline{W^{\perp}}}$ is also a unitary rep. of $G \times G$.

See e.g. Chap 3 of Sepanski, "Compact Lie groups". (GTM 235)

$\overset{!}{\Rightarrow}$ decomposable into unitary irreps V^{μ}

$\{f_j\}_{j=1}^{n_\mu}$ transforms as V^{μ} under right regular rep.

⑧

$$R(g) f_j = \sum M(g)^{\mu}_{kj} f_k$$

$$f(hg) = \sum M(g)^{\mu}_{kj} f_k(h)$$

$h=1$

$$\Rightarrow f(g) = \sum_k f_k(1) M^{\mu}_{kg}(g) \qquad (\forall g \in G)$$

$f \in W$ contradiction with the assumption $f \in W^{\perp}$

$$\Rightarrow W^{\perp} = 0$$

[if with left reg. rep.

$$L(g) f_j = \sum M^{\mu}(g)_{kj} f_k$$

$$f(g^{-1}h) = \sum M^{\mu}(g)_{kj} f_k(h)$$

$$h=1 \Rightarrow f(g) = \sum M^{\mu}(g^{-1})_{kj} f_k(1)$$

$$= \sum \overline{M^{\mu}}(g)_{jk} f_k(1)$$

$\{ \overline{M^{\mu}}_{ij} \}$ is another set of orthogonal basis]

$\Rightarrow \{ M^{\mu}_{ij} \}$ is complete.