HW.

p 20. $\overline{T(g)} \cdot \bar{v}_i = \overline{T(g) \cdot v_i}$

$$LHS = [M_{\bar{T}}(g)]_{ji} \bar{v}_j = \overline{M^*_{\bar{T}}(g)_{ji} v_j} \equiv RHS = \overline{M_T(g)_{ji} v_j} \quad \forall g$$

$$\left.\begin{array}{l} M_T(g) = M^*_{\bar{T}}(g) \\ \text{real rep.} \quad M_{\bar{T}}(g) = S M_T(g) S^{-1} \end{array}\right\} \Rightarrow M^*_T(g) = S M_T(g) S^{-1}$$

p 21. $Hom(V, W)$

$$(\tilde{T}(g) \cdot \phi)(v) = T_W(g) \cdot \phi(T_V(g^{-1}) \cdot v)$$

(1) $\tilde{T}(g_1)\tilde{T}(g_2) = \tilde{T}(g_1 g_2)$

& $\tilde{T}(e) = e$.

(2) $V^\vee := Hom(V, K)$ $\quad W = K$

$$(\tilde{T}(g) \cdot v_i^\vee)(v_j) = v_i^\vee(T(g^{-1}) \cdot v_j)$$

(3) $e_{ai}(v_j) = w_a \cdot \delta_{ij}$ $\quad$ $\{v_i\}$ V basis, $\{w_a\}$ W

$$\forall v_j : \quad [\tilde{T}(g) e_{ai}](v_j) = T_W(g) \{ e_{ai} (\sum_k [M(g)^{-1}]_{kj} v_k) \}$$

$$= T_W(g) \{ \sum_k [M(g)^{-1}]_{kj} \underbrace{e_{ai}(v_k)}_{w_a \delta_{ik}} \}$$

$$= T_W(g) [M(g)^{-1}]_{ij} w_a$$

$$= [M(g)^{-1}]_{ij} \sum_b M(g)_{ba} w_b$$

$$= \sum_b [M(g)]_{ba} [M(g)^{-1}]_{ij} e_{bj}(v_j)$$

P22 $\langle v, w\rangle_2 = \int dg \, \langle T(g) v, T(g) w\rangle_1$

$$\langle v, w\rangle = \frac{1}{|G|} \sum_g \langle T(g) v, \; T(g) w\rangle$$

$\hookrightarrow \langle T(g) v, T(g) w\rangle = \langle v, w\rangle$

Recap. Schur's lemma.

1. V_1, V_2 irrep over $\underline{k}$

   $A: V_1 \longrightarrow V_2$ an intertwiner

$$\begin{tikzcd} V_1 \arrow[r, "A"] \arrow[d, "T_1"'] & V_2 \arrow[d, "T_2"] \\ V_1 \arrow[r] & V_2 \end{tikzcd} \qquad T_2 A = A T_1 \quad \forall g.$$

   $\Rightarrow A$ isomorphism or 0.

2. (T, V) an irrep over $\underline{\mathbb{C}}$. $A: V \to V$.

   $AT = TA$

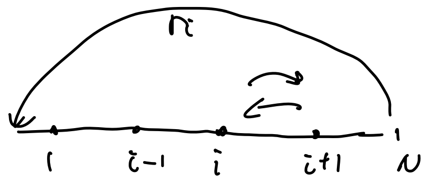
   $A(v) = \lambda v. \qquad \lambda \in \mathbb{C}.$

   $SO(2)$ on $\mathbb{R}$ as a counter example on $\mathbb{R}$

$\mathcal{H} \cong \oplus_\mu \mathcal{H}^{(\mu)}$ $\qquad$ $\mathcal{H}^{(\mu)} := \underbrace{\boxed{D_\mu}}_{A} \otimes \underline{V^{(\mu)}} \cong \oplus^{a_\mu} V^{(\mu)}$

$$T(g) = \mathbb{1} \otimes T^{(\mu)}(g)$$

$$\hat{H} = -t\left(\sum_i |i\rangle\langle i+1| + h.c.\right)$$



$\{|i\rangle\}$ $\qquad$ $|1\rangle\langle N|$

$$\begin{pmatrix} 0 & -1 & & -1 \\ -1 & 0 & \ddots & \\ & \ddots & \ddots & \\ -1 & & & \end{pmatrix} \qquad \underline{|k\rangle = \frac{1}{\sqrt{N}} \sum e^{ikr_i} |i\rangle}$$

$$\downarrow$$

$$H_k = \begin{pmatrix} \cos k & \\ & \ddots \end{pmatrix}$$

H doesn't change particle number

spin.

1 $\quad$ 2

$\left\{ \begin{array}{l} N = 0 \\ S = 0 \end{array} \right.$

$\left\{ \begin{array}{l} |2,\downarrow\rangle \otimes |2,\uparrow\rangle, \\ \frac{1}{\sqrt{2}}\left(|1,\downarrow\rangle \otimes |2,\uparrow\rangle - |1,\uparrow\rangle \otimes |2,\downarrow\rangle\right). \end{array} \right.$

$|1,\downarrow\rangle \otimes |1,\uparrow\rangle$

$\hat{N}|\varphi\rangle = n|\varphi\rangle$

2.

$$\oplus^{a_\mu} V^{(\mu)} \cong K^{a_\mu} \otimes V^{(\mu)} =: a_\mu V^{(\mu)}$$

- Pontryagin dual

Abelian group S

$\hat{S} := \mathrm{Hom}(S, U(1))$ $\qquad \chi \in \hat{S}$

$(\chi_1 \cdot \chi_2)(s) := \chi_1(s) \cdot \chi_2(s)$

$\hat{\hat{S}} := \mathrm{Hom}(\hat{S}, U(1))$

$$\hat{S} \longrightarrow U(1)$$

$$ev_s : \chi \longmapsto \chi(s) \qquad (s \in S)$$

$$\begin{aligned}(ev_{s_1} \cdot ev_{s_2})(\chi) &= ev_{s_1}(\chi) \cdot ev_{s_2}(\chi) \quad (\forall \chi \in \hat{S}) \\ &= \chi(s_1) \cdot \chi(s_2) \\ &= \chi(s_1 s_2) \\ &= ev_{s_1 s_2}(\chi)\end{aligned}$$

$\Rightarrow ev_{s_1} \cdot ev_{s_2} = ev_{s s_2}$

Theorem. (Pontryagin – van Kampen duality)

G is locally compact Abelian, then

$$\hat{\hat{S}} \cong S.$$

$$\left(\begin{array}{ccc} S & \longrightarrow & \hat{\hat{S}} \\ s & \longmapsto & ev_s \end{array} \quad \text{is an isomorphism} \right)$$

Examples

1. $S = \mathbb{Z}/n\mathbb{Z}$ $(= \mathbb{Z}_n)$

$$\chi : S \longrightarrow U(1)$$

$$\left.\begin{aligned} &\chi(\bar{1}) = \omega \\ &\chi(\bar{n}) = \chi(\bar{0}) = \omega^n = 1 \end{aligned}\right\} \quad \omega_k = e^{i\frac{2\pi}{n}\cdot k}$$

$$\underline{\chi_{\omega_k}(\bar{l}) = \omega_k^{l}}$$

$$\hat{\mathbb{Z}}_n = \{ \chi_{\omega_k} \mid \underline{\omega_k = e^{i\frac{2\pi}{n}\cdot k}} \ , k = 1, \cdots n \}$$

$$\begin{aligned} (\chi_{\underline{\omega_{k_1}}} \cdot \chi_{\underline{\omega_{k_2}}})(\bar{l}) &= \chi_{\omega_{k_1}}(\bar{l})\, \chi_{\omega_{k_2}}(\bar{l}) \\ &= (\omega_{k_1})^{\bar{l}} (\omega_{k_2})^{\bar{l}} \\ &= (\omega_{k_1} \omega_{k_2})^{\bar{l}} \\ &= \chi_{\underline{\omega_{k_1}\omega_{k_2}}}(\bar{l}) \end{aligned}$$

$$\underline{\hat{\mathbb{Z}}_n} \cong \mu_n \cong \underline{\mathbb{Z}_n}$$ Pontryagin self-dual

$$\underline{\hat{\hat{\mathbb{Z}}}_n} \cong \hat{\mathbb{Z}}_n \cong \underline{\mathbb{Z}_n}$$ (PvK theorem)

2. $S = (\mathbb{R}, +)$

$$\chi(x + y) = \underline{\underline{\chi(x+y) = \chi(x)\chi(y)}} \Rightarrow \chi(x) = e^{ax} \in U(1)$$

$\Rightarrow \chi_k(x) = e^{ikx} \in U(1)$ $\quad k \in \mathbb{R}$ $\quad x \in \mathbb{R}$

$(\chi_k \cdot \chi_\ell)(x) = e^{i(k+\ell)x} = \chi_{k+\ell}(x)$ $\quad \forall x \in \mathbb{R}$

$\hat{\mathbb{R}} \cong \mathbb{R}$ $\quad (\hat{\mathbb{R}}^n \cong \mathbb{R}^n)$ $\quad$ self-dual

$\hat{\hat{\mathbb{R}}} \cong \hat{\mathbb{R}} \cong \mathbb{R}$

3. $G = (\mathbb{Z}, +)$ $\quad \chi \in \mathrm{Hom}(\mathbb{Z}, U(1))$

$\mathbb{Z} = \langle 1 \rangle$

$\chi(1) = \xi$ $\quad \xi \in U(1)$ $\quad \xi^n = e^{ik\cdot n}$ $\quad n \in \mathbb{Z}.$

$\chi(n) = \xi^n$ $\quad \xi_1 \cdot \xi_2 = e^{i(k_1+k_2)n}$

$(\chi_{\xi_1} \chi_{\xi_2})(n) = \chi_{\xi_1}(n)\, \chi_{\xi_2}(n) = (\xi_1 \xi_2)^n = \chi_{\xi_1 \xi_2}(n)$

$\hat{\mathbb{Z}} \cong U(1)$

$$\boxed{\text{“}\chi_{k_1} \cdot \chi_{k_2} = \chi_{k_1+k_2}\text{”} \qquad e^{ik_1 n} \cdot e^{ik_2 n} = e^{i(k_1+k_2)n}}$$

(discrete $\underset{\longleftarrow}{\overset{\text{dual}}{\longrightarrow}}$ compact)

$\chi(n) = e^{ikn}$
$= e^{i(k+2\pi)n}$

$\mathbb{R}/\sim \;\cong U(1)$

4. $G = U(1) = \{e^{i\phi},\ \phi \in [0, 2\pi)\}$

$\chi(\phi + 2\pi\mathbb{Z}) = \exp[ik(\phi + 2\pi\mathbb{Z})]$

$1 = \chi(2\pi\mathbb{Z}) = \exp(ik2\pi n)$ $\quad (\forall n \in \mathbb{Z})$

$\Rightarrow k \in \mathbb{Z}$

$$\chi(\phi + 2\pi\mathbb{Z}) = \underline{\exp(i \cdot 2\pi n \phi)}$$

$$(\chi_{n_1} \chi_{n_2})(\phi) = \chi_{n_1}(\phi)\, \chi_{n_2}(\phi) = \chi_{n_1+n_2}(\phi)$$

$$\Rightarrow \widehat{U(1)} \cong \mathbb{Z}$$

$$\left.\begin{array}{l} \hat{\mathbb{Z}} \cong U(1) \\ \widehat{U(1)} \cong \mathbb{Z} \end{array}\right\} \quad \begin{array}{l} \hat{\widehat{U(1)}} \cong \hat{\mathbb{Z}} \cong U(1) \\ \hat{\hat{\mathbb{Z}}} \cong \widehat{U(1)} \cong \mathbb{Z} \end{array} \quad (\text{Pvk. } \checkmark)$$

- <u>Pontryagin dual and Fourier transform.</u>

$f \in L^1(G)$, then we define the Fourier transform

$$f : G \to \mathbb{C}$$

$$\hat{f}(\chi) = \underline{\int_G dg}\, f(x)\, \overline{\chi(x)}$$

$$\hat{f} : \hat{G} \to \mathbb{C}.$$

The inverse FT

$$f(x) = \int_{\hat{G}} d\hat{g}\, \hat{f}(\chi)\, \chi(x)$$

① FT: $f : \mathbb{R} \to \mathbb{C}$ $\quad \hat{f} : \hat{\mathbb{R}} = \mathbb{R} \to \mathbb{C}$

$$\int dx\, f(x) \to \underline{\int dx}\, f(x+a) \quad \forall a \in \mathbb{R}$$

$$\chi_k(x) = e^{ikx}$$

$$\begin{cases} \hat{f}(k) = \int_{\mathbb{R}} dx\, e^{-ikx} f(x) \\ f(x) = \int_{\mathbb{R}} dk\, e^{ikx} \hat{f}(k) \end{cases}$$

$e^{i\phi}$ $\phi \in [0, 2\pi)$ ①
$[-\pi, \pi)$

② Fourier series. $f : U(1) \to \mathbb{C}$

$\hat{f} : \mathbb{Z} \to \mathbb{C}$

$\chi_n(\phi) = e^{i 2\pi n \phi}$ $\quad (\chi \in \widehat{U(1)})$

$$\hat{f}(n) = \int_{-\pi}^{\pi} d\phi \, f(\phi) \, e^{-i \cdot 2\pi n \phi}$$

$$f(\phi) = \sum_{n \in \mathbb{Z}}' e^{i 2\pi n \phi} \hat{f}(n)$$

③ discrete FT : $f : \mathbb{Z}_n \to \mathbb{C}$ $\quad \hat{f} : \mathbb{Z}_n \to \mathbb{C}$

$\chi_k(\bar{l}) = \omega_k^l$ $\quad \chi_k \in \hat{\mathbb{Z}}_n$

$\omega_k = e^{i \frac{2\pi}{n} k}$

$$\hat{f}(k) = \sum_{l \in \mathbb{Z}_n}' f(l) \, e^{-i \frac{2\pi}{n} k \cdot l}$$

$$f(l) = \sum_{k \in \mathbb{Z}_n} \hat{f}(k) \, e^{i \frac{2\pi}{n} k \cdot l}.$$

④ discrete-time FT. $f : \mathbb{Z} \to \mathbb{C}$ $\quad \hat{f} : U(1) \to \mathbb{C}$

$\chi(z) = \zeta^z$ $\quad (z \in \mathbb{Z})$

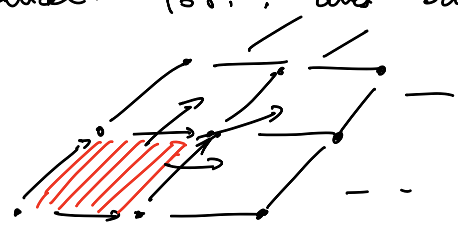
$= (e^{i\omega})^z$ $\quad (\omega \in [0, 2\pi))$

$$\hat{f}(\omega) = \sum_{n \in \mathbb{Z}} f(n) \, e^{-i\omega n}$$

$$f(n) = \int_0^{2\pi} \hat{f}(\omega) \, e^{i\omega n} \, d\omega$$

— Pontryagin dual. Tori, and band structure.

$G = \mathbb{Z}^d$



free action of $\mathbb{Z}^d$ on $\mathbb{E}^d$ $(\cong \mathbb{R}^d)$. generates a lattice $\Gamma \cong \underline{\mathbb{Z}^d}$

$(\mathbb{R}/\mathbb{Z} \cong U(1))$

The quotient $\mathbb{R}^d/\Gamma \cong \underline{U(1)^d}$

$\widehat{U(1)^d} \cong \underline{\mathbb{Z}^d} \leftarrow$ reciprocal lattice

Define the dual lattice $\Gamma^\vee \cong \mathrm{Hom}(\Gamma, \mathbb{Z})$

$\Gamma^\vee = \{ \underline{g \in \mathbb{R}^d} \mid \underline{g\gamma \in \mathbb{Z}},\ \forall \gamma \in \Gamma \} \subset \mathbb{R}^d$

reciprocal lattice vector

lattice vector

$\Gamma^\vee \cong \mathbb{Z}^d$.

$\underline{T^\vee} = \mathbb{R}^d/\underline{\Gamma^\vee}$ dual torus "Brillouin zone"

$\underline{T^\vee} \cong \mathbb{R}^d/\mathbb{Z}^d \cong U(1)^d \cong \widehat{\mathbb{Z}^d} = \underline{\widehat{\Gamma}}$

$\bar{k} \in \underline{T}^\vee$. $\underline{k = \bar{k} + g}$ $(g \in \Gamma^\vee)$

$$\chi_{\bar{k}}(\gamma) = \exp[\underline{2\pi i \cdot k \cdot \gamma}] \qquad (\gamma \in \Gamma)$$

$\bar{k}$ labels different points in BZ

corresponds to different irreps of

the translation group $\cong \mathbb{Z}^d$.