Recap. reducible & irreps

reducibility : $W \subset V$ invariant subspace

reducible rep: $W \neq 0, V$.

if not reducible : irrep.

$V = \bigoplus_i W_i$: completely reducible.

reducibility depends on the field. K

$SO(2)$ $K = \mathbb{R}$ $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ irreducible on $\mathbb{R}^2$

on $\mathbb{C}$ $\Rightarrow$ $\left(\begin{array}{c|c} e^{i\theta} & 0 \\ \hline 0 & e^{-i\theta} \end{array}\right)$ $\cong$ $U(1)$

regular rep. (of finite G)

S_3

χ	$()$	(123)	(12)
Γ_1	1	1	1
Γ_1'	1	1	-1
Γ_2	2	-1	0

$|S_3| = 6$

$\chi(e) = 6$

$\chi(g \neq e) = 0$

$V^{\text{reg. rep}} \cong V^{\Gamma_1} \oplus V^{\Gamma_1'} \oplus 2V^{\Gamma_2}$

$\Downarrow$

isotypic decomposition

isotypic components

$$V \cong \bigoplus_\mu \bigoplus_{i=1}^{a_\mu} V^{(\mu)} \qquad \underline{\underline{a_\mu}}.$$

$P: 1 \leftrightarrow 2$

$Z_2 \quad |\varphi_S\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle) \qquad P|\varphi_S\rangle = |\varphi_S\rangle$

$|\varphi_A\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle) \qquad P|\varphi_A\rangle = -|\varphi_A\rangle$

$$V \cong \underset{+}{\{|\varphi_S\rangle\}} \oplus \underset{-}{\{|\varphi_A\rangle\}}$$

– Schur's lemma

Lemm 1.

$$\begin{tikzcd} V_1 \arrow[r, "A"] \arrow[d, "T_1(g)"'] & V_2 \arrow[d, "T_2(g)"] \\ V_1 \arrow[r, "A"] & V_2 \end{tikzcd}$$

V_i irreps over any field k

$$T_2(g)A = A\,T_1(g)$$

$\Rightarrow$ $A = 0$ or isomorphism

(inj + surjective / invertible)

Proof. $\ker A := \{ v_1 \in V_1 \mid A(v_1) = 0 \}$

$\mathrm{im}\,A := \{ v_2 \in V_2 \mid \exists\, v_1 \in V_1 \text{ s.t. } v_2 = A(v_1) \}$

① $v_1 \in \ker A$. $A(T_1(g)v_1) = T_2(g)(Av_1) = 0$

$\Rightarrow T_1(g)v_1 \in \ker A$. $\forall g \in G$.

$\Rightarrow$ $\ker A$ is an invariant subspace. (of V_1)

② $v_2 \in \mathrm{im}\,A$. $T_2(g)v_2 \overset{\exists v_1 \cdot v_2 = Av_1}{=} T_2(g)A v_1$

$= \underbrace{A(T_1(g)v_1)}_{\in V_1} \in \mathrm{im}\,A$

$\Rightarrow$ $\mathrm{im}\,A$ is an invariant subspace. (of V_2)

V_1 is an irrep. $\Rightarrow$ $\ker A$ is either 0 or V

if $\ker A = V_1$. then $A = 0$ (a)

else $\ker A = 0$. A is injective (b.1)

$(\nexists\, v \neq 0.\ Av = 0)$

$\Updownarrow$

$(v_1 \neq v_2 \Rightarrow Av_1 \neq Av_2)$

$v_1 \neq v_2$ & $Av_1 = Av_2$

$\Rightarrow A(v_1 - v_2) = 0$

$\Rightarrow v_1 - v_2 \in \ker A$

$\Rightarrow$ im A is non-zero.

$\Rightarrow$ im A is V_2 $=$ A is surjective (b.2)

(b.1) + (b.2) $\Rightarrow$ A is an isomorphism.

Lemma 2. (T, V) is an irrep of G on

V $\boxed{\text{a complex vector space}}$

$A : V \to V$. intertwiner.

$\Rightarrow A(v) = \lambda v \quad (\lambda \in \mathbb{C})$

Proof. $Av = \lambda v$. $\exists\, v$ over $\mathbb{C}$.

$\Big(p(x) = \det(x\mathbb{1} - A)$ has a root in $\mathbb{C}$)

Then the eigenspace $C = \{ w : Aw = \lambda w \}$

is non-zero.

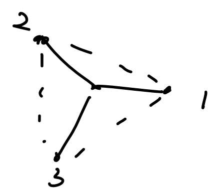
$$A\,\underline{T(g)w} = T(g)Aw = \lambda\,\underline{T(g)w} \qquad \forall g \in G.$$

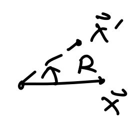
$\Rightarrow$ C is a non-trivial invariant subspace.

$\underline{V \text{ is an irrep}}$ $\Rightarrow$ $\underline{C = V}$

Γ counter example: $\mathbb{R}^2$

$$\mathbb{Z}_3 \cong A_3 \subset SO(2)$$





$R\vec{x} = \vec{x}'$

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "R"] \arrow[d, "T(g)"'] & \mathbb{R}^2 \arrow[d, "T(g)"] \\
\mathbb{R}^2 \arrow[r, "R"] & \mathbb{R}^2
\end{tikzcd}$$

$$\underline{T(g)\cdot R = R\cdot T(g)}$$

$\underline{R \text{ is an intertwiner}}$

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\neq \lambda \mathbb{1}_2 \quad \forall$$

$\Downarrow$

move to $\mathbb{C}$ $\quad R(\theta) = e^{i\theta}, e^{-i\theta} \propto 1 \; V)$

$\underline{\text{Implications for physics}}$

$\mathcal{H}$ is a Hilbert space. is a representation of some symmetry group G. and completely reducible.

$$\underline{\mathcal{H} \cong \oplus_\mu H^{(\mu)}}$$

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$$\underline{\underline{\mathcal{H}^{(\mu)} := D_\mu \otimes V^{(\mu)}}}$$

H is an Hamiltonian $H : \mathcal{H} \to \mathcal{H}$.

is an intertwiner $\underline{[H, O_G] = 0}$

$$\begin{tikzcd} \mathcal{H} \arrow[r, "H"] \arrow[d, "T(g)"'] & \mathcal{H} \arrow[d, "T(g)"] \\ \mathcal{H} \arrow[r] & \mathcal{H} \end{tikzcd}$$

Shur's lemma 2

$$\underline{H \cong \oplus_\mu \underline{H^{(\mu)}} \otimes \underline{\underline{\mathbb{1}_{V^{(\mu)}}}}}$$

Hermitian operator on D_μ

$$S H S^{-1} = \begin{pmatrix} H^{\mu_1} & & & \\ & H^{\mu_2} & & \\ & & H^{\mu_3} & \\ & & & \ddots \end{pmatrix}$$

$H^{\mu_i}_{(i, \alpha_i)}$ $\quad i = 1, \cdots n_\mu = \dim V^{(\mu)}$

$\alpha = 1 .. d_\mu$ multiplicity

$$H^{\mu}_{(i_1, \alpha_1)(i_2, \alpha_2)} = \underline{\delta_{i_1 i_2}} \, \underline{\underline{h_{\alpha_1 \alpha_2}}}$$

Schur's lemma 2

$A : V^\mu \to V^\mu \propto \mathbb{1}_{n_\mu}$

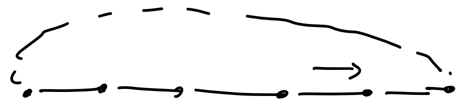
depends on the physical system

If an operator O. $[O, O_G] = 0$

$$O = \oplus_\mu O^{(\mu)} \times \mathbb{1}_{V^{(\mu)}}$$

$\langle \varphi_1, O\varphi_2 \rangle = 0$ of $\varphi_1 \in \mathcal{H}^{(\mu)}$. $\varphi_2 \in \mathcal{H}^{(\nu)}$ $(\mu \neq \nu)$.

Example



P.B.C

$$\hat{H} = -t \sum_{\langle ij \rangle} a_i^+ a_j + h.c.$$

$$G \cong C_N = \langle T \mid T^N = 1 \rangle$$

$$H/t = \begin{pmatrix} 0 & -1 & & & -1 \\ -1 & 0 & -1 & & \\ & -1 & 0 & & \\ & & & \ddots & \\ -1 & & & & 0 \end{pmatrix}$$

$N \times N$ - matrix.

basis transformation.

$$a_k^+ = \frac{1}{\sqrt{N}} \sum e^{ikr_i} a_i^+$$

$$\sum_i a_i^+ a_{i\pm1} = \cdots = 2\cos ka \, a_k^+ a_k$$

$$\tilde{H} = -2t \sum_k \cos ka \; a_k^+ a_k$$

$$-\frac{\tilde{H}}{2t} = \begin{pmatrix} \cos k_1 a & & \\ & \cos k_2 a & \\ & & \ddots \end{pmatrix} \qquad k_i = \frac{2\pi}{N} \cdot i$$

$$\mathcal{H} \cong \bigoplus_k \mathcal{H}^{(k)}$$

$$\mathcal{H}^k = \sum_i e^{ikr_i} |i\rangle$$

Example 2. multiple symmetries

— — 2-orbital system.

⑥

$$H = U \sum_m \hat{n}_{m\uparrow} \hat{n}_{m\downarrow} + (U-2J) \sum_{m \neq m'} \hat{n}_{m\uparrow} \hat{n}_{m'\downarrow}$$

$\hat{n} = d^+ d$

$$+ (U-3J) \sum_{\substack{m<m' \\ \sigma}} \hat{n}_{m\sigma} \hat{n}_{m'\sigma}$$

$$- J \sum_{m \neq m'} d^+_{m\uparrow} d_{m\downarrow} d^+_{m'\downarrow} d_{m'\uparrow}$$

$$+ J \sum_{m \neq m'} d^+_{m\uparrow} d^+_{m\downarrow} d_{m'\downarrow} d_{m'\uparrow}$$

total number conserved. $[H, N] = 0$ $N = \sum_m \hat{n}_m$

S_z conserved $[H, S_z] = 0$

↳ S^2 conserved $[H, S^2] = 0$

total global symmetry. $U(1)_N \otimes U(1)_{S_z} \subset U(1)_N \otimes SU(2)_{S^2}$

$\dim \mathcal{H} = (\mathcal{H}_1)^{\otimes 2}$ $\mathcal{H}_1 = \{ \phi, \uparrow, \downarrow, \uparrow\downarrow \}$

$\boxed{4^N}$ $\dim \mathcal{H}_1 = 4$

$= \underline{16}$

	N S		
$U(1)_N \otimes SU(2)_{S^2}$	$(0, 0)$	dim 1	
	$(1, \frac{1}{2})$	4	$\frac{\uparrow}{\downarrow} \times \frac{\uparrow}{\downarrow}$
	$(2, 0)$	$\underline{3}$	$\uparrow\downarrow$ — $\uparrow\downarrow$. $(\uparrow\downarrow - \downarrow\uparrow)$
	$(2, 1)$	$\underline{3}$	
	$(3, \frac{1}{2})$	4	
	$(4, 0)$	1	

$$\mathcal{H} = \bigoplus_{N_i, S_i} \mathcal{H}^{(N_i, S_i)}$$

$\underline{(N=2,\ S=0)}$ $\quad \begin{cases} c^\dagger_{2\downarrow} c^\dagger_{2\uparrow}, \\ \frac{1}{\sqrt{2}} (c^\dagger_{1\downarrow} c^\dagger_{2\uparrow} - c^\dagger_{1\uparrow} c^\dagger_{2\downarrow}) \\ c^\dagger_{1\downarrow} c^\dagger_{1\uparrow} \end{cases}$

$\underline{\uparrow\downarrow}\ \ \underline{\ \ }\quad U \qquad\qquad \underline{\downarrow}\ \underline{\uparrow} \rightarrow \underline{\uparrow}\ \underline{\downarrow}\quad -J$

$\underline{\uparrow}\ \ \underline{\downarrow}\quad U-2J \qquad\qquad \underline{\ \ }\ \underline{\uparrow\downarrow} \rightarrow \underline{\uparrow\downarrow}\ \underline{\ \ }\quad J$

$\underline{\uparrow}\ \ \underline{\uparrow}\quad U-3J$

$$H^{(2,0)} = \begin{pmatrix} U & 0 & J \\ 0 & \underline{U-J} & 0 \\ J & 0 & U \end{pmatrix} \rightarrow \begin{array}{ll} -J+U & \frac{1}{\sqrt{2}}(c^\dagger_{1\downarrow} c^\dagger_{1\uparrow} - c^\dagger_{2\downarrow} c^\dagger_{2\uparrow}) \\ -J+U & \\ J+U & \frac{1}{\sqrt{2}}(c^\dagger_{1\downarrow} c^\dagger_{1\uparrow} + c^\dagger_{2\downarrow} c^\dagger_{2\uparrow}) \end{array}$$

- Pontryagin duality

Abelian group $S \Longleftrightarrow U(1)$

Definition. Let S be an Abelian group.
The Pontryagin dual group $\hat{S}$.
is the group of homomorphisms
$\mathrm{Hom}(S, U(1))$. For $\chi_1, \chi_2 \in \mathrm{Hom}(S, U(1))$
define their product

$$(\chi_1 \cdot \chi_2)(s) := \chi_1(s) \cdot \chi_2(s)$$

$$(\chi_1 \cdot \chi_2)(s_1 s_2) = \chi_1(s_1 s_2) \cdot \chi_2(s_1 s_2)$$

$= \chi_1(s_1)\chi_1(s_2)\chi_2(s_1)\chi_2(s_2)$

$= [(\chi_1 \cdot \chi_2)(s_1)][(\chi_1\chi_2)(s_2)]$

$\Rightarrow \chi_1 \cdot \chi_2$ is a homomorphism.

$\hat{S} = \mathcal{H}om(S, U(1))$ is also an Abelian group

<u>Remarks</u>:

1. $\hat{S}$ is the group of all <u>complex</u> one-dimensional unitary representations of S.

2. Elements of $\hat{S}$ are called characters ($\hat{S}$ character group)