<u>Recap</u>. reducible and <u>irreducible representations</u>.

(irrep)

(T, V). $\exists$ invariant subspace W $(W \neq 0, V)$

$\Rightarrow$ W is a subrep of V.

reducible representations:

① $W \subset V$. $\begin{pmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{pmatrix}$

$V \setminus W$ is not invariant <u>indecomposable</u>

② $V \cong \bigoplus_i W_i$ $T|_{W_i}$ is a rep.

completely reducible

F.D. rep of <u>Abelian</u> groups

$\Rightarrow$ completely reducible.

$G = U(1)$

$M(z) = \mathrm{diag}\{ \rho_{n_1}(z), \rho_{n_2}(z) \cdots \rho_{n_d}(z) \}$

reducibility depends on the field.

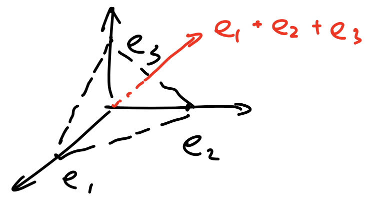
$SO(2)$. $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ irrep on $\mathbb{R}$

$\Rightarrow$ $\left(\begin{array}{c|c} e^{i\theta} & 0 \\ \hline 0 & e^{-i\theta} \end{array}\right)$ reducible on $\mathbb{C}$

Examples (cont.)

I. $S_3 \cong D_3$ on $\mathbb{R}^3 = \text{span}\{e_1, e_2, e_3\}$

$$T(\phi)\, e_i = e_{\phi(i)}$$



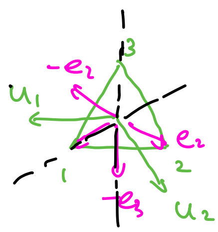
① $u_0 = e_1 + e_2 + e_3$

$$T(\phi)\, u_0 = \sum_i e_{\phi(i)} = u_0$$

$W = \text{span}\{u_0\}$ invariant subspace.

② $V \setminus W$. $W^\perp = \text{span}\{u_1, u_2\}$

a. $\begin{cases} u_1 = e_1 - e_2 \\ u_2 = e_2 - e_3 \end{cases}$ $\quad u_0 \cdot u_1 = e_1^2 - e_2^2 = 0$



$$\begin{cases} T((12))\, u_1 = -u_1 \\ T((12))\, u_2 = u_1 + u_2 \end{cases}$$

$(u_1, u_2)\, M((12)) = (-u_1, u_1 + u_2)$

$$M((12)) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\begin{cases} T((23))\, u_1 = u_1 + u_2 \\ T((23))\, u_2 = -u_2 \end{cases} \qquad M((23)) = \underline{\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}}$$

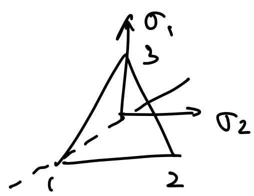
$$M((13)) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$M((23))M((13)) = M((123)) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \qquad \chi = -1$$

unitary representation w.r.t. non-ON is not a unitary matrix

b.



$$T[(23)]\sigma_1 = -\tfrac{1}{2}\sigma_1 + \tfrac{\sqrt{3}}{2}\sigma_2$$

$$T[(23)]\sigma_2 = \tfrac{\sqrt{3}}{2}\sigma_1 + \tfrac{1}{2}\sigma_2$$

$$M[(23)] = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \qquad \chi = 0$$

$$\underline{\mathbb{R}^3 \overset{u}{=} W \oplus W^\perp}$$

6. $S_3 \longrightarrow S_n$

$u_0 = \Sigma e_i$ invariant subspace.

$L = \{ x \underline{\textstyle\sum_i e_i} ,\ x \in R \}$

$L^\perp = \{ \underline{\textstyle\sum_i x_i e_i} \mid \textstyle\sum_i x_i = 0,\ x_i \in R \}$

$\Rightarrow$ Both L and $L^\perp$ are irreducible

$V \overset{u}{=} L \oplus L^\perp$

Proof that $L^\perp$ is irreducible.

if $\exists\, U \subset L^\perp$ invariant subspace

$u = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n \in U$ $\quad \underline{\Sigma x_i = 0}$

WLOG. assume $x_1 \neq x_2$ (if all x_i equal. then $u=0$)

$u - \tau(12)u = (x_1 - x_2)(e_1 - e_2) \in U$

$\Rightarrow e_1 - e_2 \in U$

$u = x_1 \cdot e_2 + x_2 \cdot e_3 + \cdots$ act $(1 2 3 \cdots n)$ on u.

$\Rightarrow e_i - e_{i+1} \in U$

$\Rightarrow \dim U \geq n-1$ & $U \subset L^\perp$ $\dim U \leq n-1$

$\Rightarrow \dim U = n-1$

$U = L^\perp$ $\quad \lrcorner$

7. examples of indecomposable reps.

a. $u(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ $\quad x \in \mathbb{R}, \mathbb{C}$.

$u(x)\, u(y) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix}$

$\left\{ \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \right\}$ is an invariant subspace

$\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \right)$

⑥

b. $B(y) = \left\{ \begin{pmatrix} \cosh y & \sinh y \\ \sinh y & \cosh y \end{pmatrix} \middle| -\infty < y < \infty \right\}$

$B(y) = \exp\left(y \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$

$T(B(y)) = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$

$B(y_1) B(y_2) = B(y_1 + y_2)$

$\Rightarrow T(y_1) T(y_2) = T(y_1 + y_2)$

c. $A \in GL(n, k)$

$T(A) = \begin{pmatrix} 1 & \log|\det A| \\ 0 & 1 \end{pmatrix}$

$T(A) T(B) = \begin{pmatrix} 1 & \log|\det A| + \log|\det B| \\ 0 & 1 \end{pmatrix}$

$= T(AB)$

d. symorphic Space groups

$T \rtimes_R PG$. semidirect product

$R \in O(3)$

$\vec{\tau} \in T$ $\quad \{R | \vec{\tau}\} \in$ Euclidean group

$\underline{\{R_1 | \vec{\tau}_1\}} \underline{\{R_2 | \vec{\tau}_2\}} \vec{r} = \{R_1 | \vec{\tau}_1\} (R_2 \vec{r} + \vec{\tau}_2)$

$= R_1 R_2 \vec{r} + (R_1 \vec{\tau}_2 + \vec{\tau}_1)$

$= \{\underline{R_1 R_2} | \underline{\underline{R_1 \vec{\tau}_2 + \vec{\tau}_1}}\} \vec{r}$

$(\vec{\tau}_1, R_1) \circ (\vec{\tau}_2, R_2) = (\underline{\vec{\tau}_1 + R_1\vec{\tau}_2}, R_1 R_2)$

Matrix rep. $\begin{array}{cc} 3 & 1 \\ \end{array}$ $\begin{pmatrix} \boxed{R} & \tau \\ 0 & 1 \end{pmatrix}$

<u>Proposition</u>. Let (T, V) be a <u>unitary rep.</u> of an inner product space V. and $W \subset V$ is an invariant subspace. Then $W^\perp$ is an invariant subspace. ($W^\perp = \{ y \in V \mid \langle y, x \rangle = 0, \forall x \in W \}$)

<u>Proof</u>. $\forall g \in G$, $y \in W^\perp$:

$$\langle T(g) y, x \rangle = \cancel{\langle y, T(g)^{+} x \rangle}$$
$$= \langle \underbrace{y}_{\in W^\perp}, \underbrace{T(g^{-1}) x}_{\in W} \rangle$$
$$= 0$$

$\Rightarrow T(g) y \in W^\perp$

$\Rightarrow W^\perp$ invariant subspace.

Corollaries:

1. F.D. unitary rep. are always completely reducible.

V reducible $\Rightarrow$ $V = W \oplus W^{\perp}$

? $W' \oplus W'^{\perp}$

$V = \oplus W_i$

2. For compact groups. reps are f.d. unitarizable.

$\Rightarrow$ completely reducible

3. Finite G. Regular rep. $L^2(G)$ is completely reducible.

($L_g \cdot \delta_h = \delta_{gh}$ δ-basis $\delta_g(h) = \begin{cases} 1 & h=g \\ 0 & \text{other} \end{cases}$)

$|G|$-dim rep.

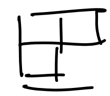
Example of reg. rep. of S_3

$\begin{cases} \chi(e) = |S_3| = 6 \\ \chi(g \neq e) = 0 \end{cases}$

$SMS^{-1} = \begin{pmatrix} \bigcirc & & \\ & \bigcirc & \\ & & \bigcirc \end{pmatrix}$

conj. class

character		$()$	(123)	(12)
trivial	ρ_1	1	1	1
sgn	ρ_1'	1	1	-1
	ρ_2	2	-1	0



$$V^{reg.} = x\,\rho_1 \oplus y\,\rho_1' \oplus z\,\rho_2$$

$$x + y + 2z = 6$$

$$\begin{cases} x + y - z = 0 \\ x - y + 0 \cdot z = 0 \end{cases} \Rightarrow \begin{cases} x = y = 1 \\ z = 2 \end{cases}$$

$$V^{reg.} = V^{\rho_1} \oplus V^{\rho_1'} \oplus 2V^{\rho_2}$$

- Isotypic components

Assume that the set of irreps. (up to isomorphism) of G is countable

choose a representative $(T^{(\mu)}, V^{(\mu)})$ for each isomorphism class

$$V \cong \bigoplus_\mu \boxed{\bigoplus_{i=1}^{a_\mu} V^{(\mu)}}$$

a_μ is the number of times $V^{(\mu)}$ appears in the decomposition.

$\bigoplus_{i=1}^{a_\mu} V^{(\mu)}$ is the isotypic component of V associated to μ.

We can identify

$$V^{(\mu)} \oplus V^{(\mu)} \oplus \cdots \oplus V^{(\mu)} \cong \underline{K^{a_\mu} \otimes V^{(\mu)}} =: a_\mu V^{(\mu)}$$

$$\circlearrowleft\ T(g) = \mathbb{1}_{a_\mu} \otimes T(g)^{(\mu)}$$

<u>Example</u>. rep Z_2 on a vector space (as linear operators)

$$T: V \to V \qquad T \in \mathrm{Hom}(V, V)$$

$$T^2 = \mathbb{1}$$

projector: $P_\pm = \frac{1}{2}(1 \pm T)$

$$\underline{V^+} := \underline{\ker(P_+)} = \{ v \mid P_+ v = 0 \} = \underline{P_-(V)}$$
$$Tv = -v$$

$\underline{-1 \text{ eigenspace}}$ of T

$$V^- = \ker(P_-) = \{ v \mid P_- v = 0 \} = P_+(V)$$
$$Tv = v$$

$\underline{+1 \text{ eigenspace}}$

Z_2 has two 1D irreps. $\rho_+(1) = \rho_+(\sigma) = 1$
$\{1, \sigma\}$

$$\begin{cases} \rho_-(1) = 1 \\ \rho_-(\sigma) = -1 \end{cases}$$

$$\underline{V \cong \overset{P_-(V)}{V^+} \otimes \rho_- \oplus \overset{P_+(V)}{V^-} \otimes \rho_+}$$

$$\underline{V = \mathbb{R}^2} \qquad T(\sigma) = \underline{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \cong \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -1 \end{array}\right) \qquad V = \underline{\rho_+} \oplus \underline{\rho_-}$$

$V = \mathrm{span}\{|1\rangle, |2\rangle\}$ $\quad |\tilde{1}\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle) \quad \sigma|\hat{1}\rangle = |\hat{1}\rangle$

$|\tilde{-1}\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle) \quad \sigma|\hat{-1}\rangle = -|\hat{-1}\rangle$

- Schur's lemma

recall an intertwiner A is a morphism of G-spaces

$$\begin{tikzcd} V_1 \arrow[r, "A"] \arrow[d, "T_1(g)"'] & V_2 \arrow[d, "T_2(g)"] \\ V_1 \arrow[r, "A"] & V_2 \end{tikzcd}$$

$$T_2(g) A = A T_1(g) \quad \left(\not\Rightarrow T_2(g) = A T_1(g) A^{-1} \right)$$

A might be 0.

Lemma 1. Let G be any group. Let V_1, V_2 be vector spaces over any field K. s.t. they are carrier spaces of irreps of G.

If $A: V_1 \longrightarrow V_2$ is an intertwiner between these irreps. then A is either zero or an isomorphism.

Lemma 2: Suppose (T, V) is an irrep of G on a complex vector space V by linear transformation, and $A: V \to V$ is a $\mathbb{C}$-linear intertwiner

$$(AT(g) = T(g)A, \ \forall g \in G)$$

Then A is proportional to the identity transformation

$$A(v) = \lambda v. \quad (\lambda \in \mathbb{C}, \ \forall v \in V).$$

$$\left(\mathrm{End}_{\mathbb{C}}^{G}(V) \cong \mathbb{C} \right)$$