Recap.

$$\int_G f(g)\,dg = \int_G f(hg)\,dg \quad \text{Haar measure}$$

$$\hookrightarrow \quad L^2(G) = \{ f : G \to \mathbb{C} \mid \underbrace{\int |f(g)|^2 dg}_{\substack{\shortparallel \\ \langle f, f\rangle}} < \infty \}$$

$$\int |f(g)|^2 dg = \int |f(h_1 g h_2^{-1})|^2 dg$$

$G \times G$ action on G:

$$(g_1, g_2) \longmapsto L(g_1) L(g_2^{-1})$$

$$(g_1, g_2) \cdot g_0 = g_1 g_0 g_2^{-1}$$

$\hookrightarrow$ induced action on $Map(G, \mathbb{C})$

$$[(g_1, g_2) \cdot f](h) := f(g_1^{-1} h g_2)$$

(T, V)

$$(g_1, g_2) \cdot S = T(g_1) \cdot S \cdot T(g_2)^{-1} \qquad S \in End(V)$$

$$End(V) \longrightarrow L^2(G)$$

$$S \longmapsto f_S$$

$$\underline{\underline{f_S := Tr_V(S\,T(g^{-1}))}}$$

$$\begin{array}{ccc} End(V) & \xrightarrow{\iota} & Map(G, \mathbb{C}) \\ \downarrow T_{End(V)} & & \downarrow T_{reg.\ rep} \\ End(V) & \longrightarrow & Map(G, \mathbb{C}) \end{array}$$

take s as matrix unit e_{ij}

$\Rightarrow$ $f_s = M_{ji}(g^{-1})$

Example. $G = \{1, \omega, \cancel{\omega^3}\ \omega^2\}$

$\boxed{\delta_j(\omega^k)} = \begin{cases} 1 & j = k \bmod 3 \\ 0 & \text{else} \end{cases}$

$\left(\frac{1}{|G|} \sum_g \delta_i(g)\delta_j(g) = \delta_{i,j} \right)$

$(L(\omega)\delta_0)(g) = \delta_0(\omega^{-1}g) = \delta_1(g)$

$\delta_1 \qquad \delta_2$

$\delta_2 \qquad \delta_0$

$M(\omega) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$\{M_{i,1}\}$

$\begin{cases} M_{11}(\omega) = 0 \\ (\omega^2) \quad 0 \\ (1 = \omega^3) = 1 \end{cases}$

$M(\omega^2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

$M(\omega^3) = M(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Zee. $g \cdot e_a = e_{ga}$

①

	g_1	g_2	g_3	g_4
	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$\langle a, b \mid a^2 = b^2 = (ab)^2 = 1 \rangle$

$$\delta_{g_i}(g_j) = \begin{cases} 1 & i=j \\ 0 & \text{else} \end{cases}$$

$$(L(g_2 = a) \cdot \delta_{g_i})(g)$$
$$= \delta_{g_i}(a^{-1} g)$$
$$= \delta_{a g_i}(g)$$

$$L(a)\,\delta_e = \delta_a$$
$$L(a)\,\delta_a = \delta_e$$
$$L(a)\,\delta_b = \delta_{ab} = \delta_c$$
$$L(a)\,\delta_c = \delta_{ac} = \delta_b$$

$$M(a) = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\chi(M(g \neq e)) = 0$$
$$\chi(M(e)) = \dim V$$
$$= |G|$$

– Reducible & irreducible representations

Recall direct sum of reps

$$T_{V \oplus W} = T_V \oplus T_W$$

$$M_{V \oplus W} = \left(\begin{array}{c|c} M_V & 0 \\ \hline 0 & M_W \end{array}\right)$$

? $M \xrightarrow{?} SMS^{-1} = \begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{pmatrix} \} d$

$\Rightarrow$ we want to "reduce" a rep. of large dimensions into smaller ones.

<u>Definition</u> Let $W \subset V$ be a <u>linear subspace</u> of carrier space V. of a group rep.

$T : G \to GL(V)$, Then W is invariant under T. (W is an <u>invariant subspace</u>) if $\forall g \in G$. $w \in W$

$$T(g) \cdot w \in W$$

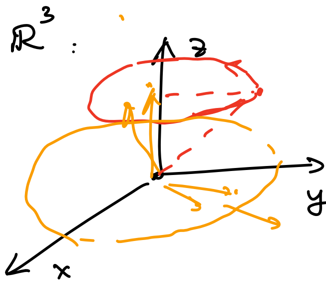
Example

1. $\{\vec{0}\}$ & V

2. $SO(2)_{\hat{z}}$ on $\mathbb{R}^3$:

$v_1, v_2 \in V$

$\Rightarrow \alpha v_1 + \beta v_2 \in V$



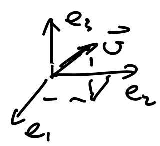
xy plane is an invariant subspace

3. canonical rep. of S_n:

$$T(\phi): \vec{e}_i \longrightarrow \vec{e}_{\phi(i)}$$

$$\vec{v} = \vec{e}_1 + \vec{e}_2 + \cdots + \vec{e}_n$$

$$T(\phi)\vec{v} = T(\phi)\sum_i \vec{e}_i = \sum_i \vec{e}_{\phi(i)} = \vec{v}$$

S_3: 

body diagonal

4. Matrix representation:

$$M : G \longrightarrow GL(n, K)$$

$$M_{ij} : G \longrightarrow K$$
$$g \longmapsto M_{ij}(g)$$

Consider the linear span of M_{ij} with fixed i

$$R_i = \text{span}\{M_{ij},\ j=1, \cdots n\} \subset L^2(G)$$

and right action:

$$(R(g)\cdot M_{ij})(h) = M_{ij}(hg) \qquad \in R_i$$

$$= \sum_s \underbrace{M_{sj}(g)}_{\in K \text{ coefficients}} \cdot M_{is}(h)$$

$\Rightarrow R_i$ is an invariant subspace

(under right action)

$\mathcal{L}_j = \text{span}\{ M_{ij} .\ i=1,\cdots n \}$

is invariant (under left action)

$\mathcal{M} = \mathcal{L}\mathcal{R} = \text{span}\{ M_{ij} ,\ i,j=1.\cdots n \} \subset L^2(G)$

is invariant under $G \times G$ - action

$((g_1, g_2)\cdot f)(h) = f(g_1^{-1} h g_2)$

If we only consider the left-regular rep.

$$\mathcal{L}\mathcal{R} \cong \bigoplus_{i=1}^{n} \mathcal{L}_i$$

every L_i is an invariant subspace of $\mathcal{L}\mathcal{R}$

$\Rightarrow L^2(G)$ is highly <u>reducible</u>.

<u>Definition</u>. A representation $(T.V)$ is <u>reducible</u>

if there is a proper, nontrivial invariant subspace $W \subset V$ ($W \neq V, 0$)

If V is not reducible, it is an <u>irreducible representation</u>. ("irrep")

Remarks.

1. $\forall v \in V, V' = \text{span}\{T(g)v, \forall g \in G\}$ is an invariant space $T(g_1)(T(g_2)v) = T(g_1 g_2)v \in V'$

   If (T, V) is an irrep. $V' = V$

   v is called a cyclic vector.

   Note that the existance of a cyclic vector does not imply that the representation is irreducible.

   canonical rep of S_n.

   $$T(\phi) : e_i \to e_{\phi(i)}$$

   e_1 is a cyclic vector.

   $\sum e_i$ is a proper, nontrivial invariant subspace.

2. (T, V) a representation. $\exists W \subset V$ an invariant subspace. We can restrict T to W. $(T|_W, W)$ is a subrepresentation of (T, V)

   $$T|_W(g) = T(g)|_W$$

   We will write T instead of $T|_W$ if

no confusion arises.

If T is unitary on V. then it is unitary on W.

$$\langle Tv_1, Tv_2\rangle = \langle v_1, v_2\rangle \quad \forall v_i (\in W) \subset V$$

3. (T, W) is subrep of (T, V)

choose an ordered basis $\{w_1, \dots w_k\}$

Then it can be completed to an ordered basis $\{w_1, \dots w_k, u_{k+1}, \dots u_n\}$

$$T(g)(w_i) = (M_{11}(g))_{ji}\, w_j + (M_{21}(g))_{ai}\, u_a$$
$$T(g)(u_a) = (M_{12}(g))_{ja}\, w_j + (M_{22}(g))_{ba}\, u_b$$

in matrix form $(\{w\}, \{u\}) \overset{M}{\nwarrow} \begin{pmatrix} \boxed{M_{11}} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$

W invariant $\Longrightarrow M_{21} = 0$.

$$\Rightarrow T(g)(w_i) = \sum_j M_{11}(g)_{ji}\, W_j$$

$\Rightarrow M_{11}$ is a representation on W.

M_{22} is not a rep on $V \setminus W$.

If we want to further simplify the representation

Define a change of basis $\begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix}$

$$(w,\ u)\begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} = (w,\ wS+u) \equiv (w,\ u')$$

$$\begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -S \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -S \\ 0 & 1 \end{pmatrix}\begin{pmatrix} M_{11}(g) & M_{12}(g) \\ 0 & M_{22}(g) \end{pmatrix} = \begin{pmatrix} M_{11}(g) & M_{12}(g) - SM_{22}(g) \\ & M_{22}(g) \end{pmatrix}$$

$$\Rightarrow \underline{M_{12}(g) = SM_{22}(g)}. \ (\forall g \in G)$$

such an S exists for <u>completely reducible</u> representations.

4. Consider the quotient space V/W

$$v_1 \sim v_2 \quad \text{iff} \quad v_1 - v_2 \in W$$

$$T(g)(v+W) = T(g)(v) + W$$

$$\begin{aligned} T(g_1)T(g_2)(v+W) &= T(g_1)(T(g_2)v + W) \\ &= T(g_1 g_2)v + W \\ &= T(g_1 g_2)(v+W) \end{aligned}$$

<u>from above</u>:

$$T(g)(\underline{u_a}) = \underline{(M_{12}(g))_{ja}} w_j + \underline{(M_{22}(g))_{ba} u_b}$$

$$\Rightarrow T(g)(u_a + W) \overset{\sim}{=} (M_{22}(g))_{ba} u_b + W.$$

$\Rightarrow M_{22}$ is the representation on the quotient space V/W.

– Complete reducibility

Definition. A representation (T, V) is called completely reducible if it is isomorphic to a direct sum of representations $W_1 \oplus W_2 \oplus \cdots \oplus W_n$

where W_i are irreps. Thus, there is a basis in which the representation matrices look like

$$M(g) = \begin{pmatrix} M_{11}(g) & 0 & \cdots \\ 0 & M_{22}(g) & 0 \\ \vdots & 0 & \ddots \; M_{33}(g) \\ & & & \ddots \end{pmatrix}$$

If a representation is reducible but not completely reducible, it is "indecomposable".

irreps are completely reducible

Examples.

1. $G = \mathbb{Z}_2 \cong S_2 = \{e, \tau\}$

$$M(e) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right) \qquad M(\tau) = \left(\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right) \quad = (T, V)$$

$$A = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \Rightarrow \hat{M}(\tau) = A^{-1} M A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\hat{M}(e) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right)$, $\hat{M}(\tau) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -1 \end{array}\right)$

$\rho_+(e) = \rho_+(\tau) = 1$ trivial rep.

$\rho_-(e) = 1$, $\rho_-(\tau) = -1$ sgn rep.

$e = ()$ $\tau = (12)$

The original rep. $(T, V = \mathbb{R}^2)$ is completely reducible. $(T, V) \cong \rho_+ \oplus \rho_-$

2. $G = U(1) = \{ z \in \mathbb{C} \mid |z| = 1 \}$. $V = \mathbb{C}$

$\rho_n(z) = z^n$ for $\forall n \in \mathbb{Z}$.

$$\rho_n(z_1 z_2) = (z_1 z_2)^n = z_1^n \cdot z_2^n = \rho_n(z_1)\rho_n(z_2)$$

inequivalent for different n.

3. Finite-dimensional representations of Abelian groups are completely reducible on $\mathbb{C}$.

(Any complex irrep of an Abelian group is one-dim.)

Choosing an orthonormal (ON) basis set.

s.t. all $M(g)$ are commuting unitary matrices over the complex field.

$$M(g_i)M(g_j) = M(g_j)M(g_i) \quad \forall g_i, g_j \in G.$$

$\Rightarrow$ M's can be simultaneously diagonalized.

$$M(z) = \mathrm{diag}\{\lambda_1(z), \lambda_2(z), \cdots \lambda_n(z)\}$$ 12

$\Rightarrow G = U(1)$ any f.d. rep. on $V \cong \mathbb{C}^d$.

$$M(z) = \mathrm{diag}\{\rho_{n_1}(z), \rho_{n_2}(z), \cdots \rho_{n_d}(z)\}$$

$$V \cong \rho_{n_1} \oplus \rho_{n_2} \oplus \cdots \oplus \rho_{n_d}.$$

$SO(2)$: $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ on $\mathbb{R}^2$

$$= \left(\begin{array}{c|c} e^{i\theta} & 0 \\ \hline 0 & e^{-i\theta} \end{array}\right)$$

completely reducible on $\mathbb{C}$.

irreducible on $\mathbb{R}$.