HW. P12. $\phi_1(u) = u \qquad \phi_2(u) = u^*$

$tr(u) = tr(u^*)$

$SU(2) \quad \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \qquad |\alpha|^2 + |\beta|^2 = 1$

$tr = \alpha + \bar{\alpha} = \bar{\alpha} + \alpha$

$\tau u \tau^{-1} = u^* \qquad \tau = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

P13. $H \triangleleft G. \qquad gH = Hg$

H, gH
$\Rightarrow gH = Hg$
H, Hg

P14. $A_n \triangleleft S_n \qquad [S_n : A_n] = 2$

$\phi \in A_n \qquad \phi$: even transposition prod.

$\tau \phi \tau^{-1} \in A_n \quad \Rightarrow \tau A_n = A_n \tau$

P15 $g[a, b]g^{-1} = [gag^{-1}, gbg^{-1}] \in [G, G]$

(2) $H \triangleleft G$.

G/H is abelian $\Leftrightarrow [G, G] \subset H$

$\Rightarrow \quad (aH)(bH) = abH = (bH)(aH) = baH$

$\Rightarrow \quad abh_1 = bah_2$

$a^{-1}b^{-1}ab = h_2 h_1^{-1} \in H$

$[a^{-1}, b^{-1}] \in H$

$G / [G, G] = H_1(G)$ first homology of G.

P16. $D \subset SU(2)$ $\quad D = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \cong U(1)$

(a) $gD = Dg$ $\quad u = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$

$u d u^{-1} \in D$

$\Rightarrow \underline{\alpha = 0}.\ \underline{\beta = 0}$

$N_{SU(2)}(D) = \left\{ \begin{pmatrix} 0 & \bar{z} \\ -\bar{z} & 0 \end{pmatrix}, |z| = 1 \right\} \cup = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} D$

$\left\{ \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}, |z| = 1 \right\} \Rightarrow D$

$= D \cup \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} D$

$\underline{N_{SU(2)}(D) / D} = \left\{ \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} D \right.$

$\left. \begin{pmatrix} 0 & -\bar{z} \\ z & 0 \end{pmatrix} D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} D \right\} \underline{\cong \mathbb{Z}_2}$

$u d u^{-1} = \tilde{d}$ $\quad \tilde{d} = \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}$, $\underline{u \in D}$

$\tilde{d} = \begin{pmatrix} \bar{z} & 0 \\ 0 & z \end{pmatrix}$ $\quad u \in \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} D$

(d) $N_{SU(2)}(D)$ $\quad \left\{ \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right\}, \left\{ \begin{matrix} 0 & z \\ -\bar{z} & 0 \end{matrix} \right\}, \left\{ \begin{matrix} -1 & 0 \\ 0 & -1 \end{matrix} \right\}$

P17. effective group action $\phi : G \to S_X$

(a) effective: $\forall g \neq 1 \quad \exists x . \; gx \neq x$

$\Rightarrow \quad \phi(g)$ is nontrivial $\quad \phi(g) \neq 1$

homo. inj. $\Rightarrow \quad \phi(g) = 1$ iff $g = 1$

(b) $\begin{array}{l} g_1 \cdot x = x \\ g_2 \cdot x = x \end{array} \Rightarrow (g_1 \cdot g_2) x = x$

$\forall g_{i,j} \in H \Rightarrow g_i g_j \in H$

$\begin{array}{l} g_i (g \cdot x) = g_i x' = x' \\ g (g_i \cdot x) = g \cdot x = x' \end{array} \quad \Rightarrow \forall g \quad \underline{g_i g = g g_i} \quad \underline{g_i \in H}$

(c) $G/H \times X \to X$

$(gH) \cdot x := g \cdot x$

$\forall x \in X . \quad (gH) x = x$

$\Leftrightarrow gx = x$

$\Leftrightarrow g \in H$

$\Leftrightarrow gH = H = \underline{1_{G/H}}$

18 Burnside's lemma. $|\# \text{orb}| = \frac{1}{|G|} \sum_{g \in G} |X^g|$

transitive $\Longleftrightarrow$ one orbit. $\forall x, y . \exists g$. s.t $y = gx$.

$|G| = \sum_{g \in G} |X^g|$ ⊛

$\sum_{g \in G} |X^g| \geq \sum_{g \in G} 1 = |G|$

$|X^e| = |X| > 1$

$\exists g. \; |X^g| = 0$

Recap **Haar measure**

$$\frac{1}{|G|}\sum_{g\in G} f(g) = \frac{1}{|G|}\sum_{g\in G} f(hg) \longrightarrow \int_G dg$$

$$\underline{\int_G f(hg)\,dg = \int_G f(g)\,dg} \quad (\forall h \in G)$$

left invariance.

compact / finite left = right up to scale

locally compact left $\neq$ right

$$\int_G f(g)\, d(h^{-1}g) = \int_G f(g)\,dg \Rightarrow \underline{d(h^{-1}g) = dg}$$

$G = \mathbb{R}^*_{>0}$

$$\int f(g)\,dg = \qquad \frac{dx}{x} \qquad \underline{\frac{d(x/a)}{x/a} = \frac{dx}{x}}$$

$$g \longmapsto g' = g_0 g$$

$$\prod_{ij} dg'_{ij} \longmapsto \left| \frac{\partial(g'_{11}, g'_{12}, \cdots g'_{nn})}{\partial(g_{11} \cdots g_{nn})} \right| \prod_{ij} dg_{ij}$$

$$g'_{ij} = \sum_k (g_0)_{ik} g_{kj}$$

$$\frac{\partial g'_{ij}}{\partial g_{kl}} = (g_0)_{ik}\underline{\delta_{jl}}$$

$\forall_{j,l=1}$

$$\begin{pmatrix} g_0 & & \\ & g_0 & \\ & & g_0 \end{pmatrix}$$

$\underline{|\det g_0|^n}$

(T, V) a rep on an inner product space

$$\langle v, w\rangle_2 := \underline{\int_G \langle T(g)v, T(g)w\rangle_1 \, dg}$$

$$\langle T(h)v, T(h)w\rangle_2 = \int_G \langle T(hg)v, T(hg)w\rangle_1 \, dg$$

$$= \langle v, w\rangle_2$$

$\hookrightarrow \quad H = \sum_g T(g)^\dagger T(g) \qquad$ finite group.

8.6. Regular representation

Let G be a group. Then there is a left action of $G \times G$ on G.

$$(g_1, g_2) \longmapsto L(g_1) R(g_2^{-1})$$

$$(g_1, g_2) \cdot g_0 = g_1 g_0 g_2^{-1}$$

restrict to $G \times \{1\}$. $\{1\} \times G$.

$$(g_1, 1) \cdot g_0 = g_1 g_0 \qquad (1, g_2) \cdot g_0 = g_0 g_2^{-1}$$

There is an associated induced action on $\underline{Map(G, \mathbb{C})}$

$$[\underline{(g_1, g_2) \cdot f}](h) := f(g_1^{-1} h g_2)$$
$$=: \underline{f}'(h)$$

$\Rightarrow$ the vector space $Map(G, \mathbb{C}) = \{ f : G \to \mathbb{C} \}$ becomes a representation space of $G \times G$.

$$\left([(g_1, g_2)(g_3, g_4) f](h) = \{ (g_1, g_2) [(g_3, g_4) f] \}(h) \right)$$

$$G \times G \longrightarrow Hom(\{f\}, \{f\}) =: End(\{f\})$$

(Endmorphism: $\varphi : V \to V$

End + iso = $Aut(V)$)

Now, equip G with a left and right invariant Haar measure, and consider the Hilbert space

$$L^2(G) = \{ f : G \to \mathbb{C} \mid \underbrace{\int |f(g)|^2 dg}_{\langle f, f \rangle} < \infty \}$$

(in physics $\int |\varphi(x)|^2 dx = 1 \ (< \infty)$

$\int \overline{\varphi(x)}\, \phi(x)\, dx \le \sqrt{\int |\varphi(x)|^2 dx \int |\phi(x)|^2 dx} < \infty$)

Then $G \times G$ action preserves the L^2-property and it is unitary. (because of the Haar measure)

$$\int |f(g)|^2 dg = \int |f(h_1 g h_2^{-1})|^2 dg$$

Definition. The representation $L^2(G)$ is known as the regular representation of G.

If we restrict $G \times G$ to subgroups $G \times \{1\}$ or $\{1\} \times G$, then

$$(L(h) \cdot f)(g) := f(h^{-1} g)$$

is the left regular representation

$$(R(h) \cdot f)(g) = f(gh)$$

the right regular representation.

Suppose (T, V) is a representation of G.

We can define $G \times G$ action on $\mathrm{End}(V) = \mathrm{Hom}(V, V)$

$\forall S \in \mathrm{End}(V)$

$$(g_1, g_2) \cdot S = T(g_1) \cdot S \cdot T(g_2)^{-1}$$

For finite-dimensional V. we can define a map.

$$\iota : \mathrm{End}(V) \longrightarrow L^2(G)$$
$$S \longmapsto f_S$$
$$f_S := \mathrm{Tr}_V(S\, T(g^{-1}))$$

which is equivariant (ι is an intertwiner)

$$\begin{tikzcd}
\mathrm{End}(V) \arrow[r, "\iota"] \arrow[d, "T_{\mathrm{End}(V)}"] & \mathrm{Map}(G, \mathbb{C}) \arrow[d, "T_{\mathrm{reg.rep.}}"] \\
\mathrm{End}(V) \arrow[r] & \mathrm{Map}(G, \mathbb{C})
\end{tikzcd}$$

$$\begin{tikzcd}
S \arrow[r] \arrow[d] & f_S \arrow[d] \\
(h_1, h_2) S \arrow[r] & \boxed{\begin{matrix} (h_1, h_2) f_S \\ \parallel \\ f_{(h_1, h_2) S} \end{matrix}}
\end{tikzcd}$$

$$(h_1, h_2) f_S = f_S(h_1^{-1} g h_2)$$
$$= \mathrm{Tr}_V(S\,T(h_2^{-1} g^{-1} h_1))$$
$$= \mathrm{Tr}_V(S\,T(h_2)^{-1}\,T(g^{-1})\,T(h_1))$$
$$= \mathrm{Tr}_V(\underline{(h_1, h_2) S}\,T(g^{-1}))$$
$$= f_{(h_1,h_2)S}(g)$$

Equip V with an ordered basis $\{v_i\}$

$$T(g)\cdot v_i = \sum_j M(g)_{ji}\, v_j$$

and take S to be the basis of $\mathrm{End}(V)$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ & \vdots & & \end{pmatrix}$$

matrix unit $\underline{e_{ij}}$, $[e_{ij}]_{ab} = \delta_{ia}\delta_{jb}$ $\begin{cases} 1 & \text{on } (i,j) \\ 0 & \text{otherwise} \end{cases}$

$$f_S = \mathrm{Tr}_V(S\,T(g^{-1}))$$
$$= \mathrm{Tr}\Big(\sum_b \delta_{ia}\delta_{jb} M_{bc}(g^{-1})\Big)$$
$$= \sum_{ac} [\delta_{ia} M_{jc}(g^{-1})]\,\delta_{ac}$$
$$= \underline{M_{ji}(g^{-1})}$$

(if replace V by its dual space $V^\vee$

$M^\vee(g) = [M(g^{-1})]^{tr} = M(g)^{tr,-1}$ (last lecture)

$f_S = M_{ij}(g)$)

$\Rightarrow$ f_s's are linear combinations of matrix elements of rep. of G.

Example. 1. $G = \mu_3 = \{1, \omega, \omega^3\}$

$$\delta_j(\omega^k) = \begin{cases} 1 & j = k \bmod 3 \\ 0 & \text{else} \end{cases}$$

$$(L(\omega)\cdot\delta_0)(g) = \delta_0(\omega^{-1}\underline{g}) = \underline{\underline{\delta_1(g)}}$$

$$\begin{cases} L(\omega)\delta_0 = \delta_1 \\ L(\omega)\delta_1 = \delta_2 \\ L(\omega)\delta_2 = \delta_0 \end{cases} \qquad L(\omega) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$