Recap. 1. intertwiner / intertwing map

$$\begin{tikzcd} V_1 \arrow[r, "A"] \arrow[d, "T_1(g)"'] & V_2 \arrow[d, "T_2(g)"] \\ V_1 \arrow[r] & V_2 \end{tikzcd}$$

LHS

RHS

$Hom_G(V_1, V_2)$

$$T_2(g) A = A T_1(g)$$

2. $(T_1, V_1) \cong (T_2, V_2)$

$$T_2(g) = A T_1(g) A^{-1} \quad (\forall g \in G)$$

3. unitarizable rep. is equivalent to a unitary rep.

$$\langle U(g) v, U(g) w \rangle = \langle v, w \rangle \quad \forall v, w \in V$$

8.2 unitary representations (cont.)

Consider a finite group. Let $T(g)$ be a (non-unitary) rep. To unitarize $T(g)$, define

$$H = \sum_{g \in G} T^{\dagger}(g)\, T(g)$$

H is Hermitian and positive definite.

$$T^{\dagger}(h)\, H\, T(h) = \sum_{g} T^{\dagger}(h)\, T^{\dagger}(g)\, T(g)\, T(h) = \sum_{g} T^{\dagger}(gh)\, T(gh) = H$$

$$\exists U \text{ s.t. } U^{\dagger} H U = \Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n) \quad (\forall \lambda_i > 0)$$

Define $\tilde{T}(g) = \Lambda^{\frac{1}{2}} U^{\dagger} T(g) U \Lambda^{-\frac{1}{2}}$

$$\tilde{T}^{\dagger}(g)\, \tilde{T}(g) = (\Lambda^{-\frac{1}{2}} U^{\dagger} \underbrace{T^{\dagger}(g) \underbrace{U \Lambda^{\frac{1}{2}})(\Lambda^{\frac{1}{2}} U^{\dagger}}_{H} T(g)}_{H} U \Lambda^{-\frac{1}{2}})$$

$$= \Lambda^{-\frac{1}{2}} \underbrace{U^{\dagger} H U}_{\Lambda} \Lambda^{-\frac{1}{2}} = \mathbb{1}$$

$$\Rightarrow \tilde{T}(g) = A^{-1}\, T(g)\, A \qquad A = U \Lambda^{-\frac{1}{2}} \qquad (\forall g)$$

$\Rightarrow$ Representations of finite groups are equivalent to unitary representations

$\Rightarrow$ What about continuous / infinite groups?

Some ideas: $\sum_{g\in G} \to \int_G dg$?

$\hookrightarrow$ Haar measure (later)

8.3. Direct sum, tensor product, and dual representations

(T_1, V_1) and (T_2, V_2) are two reps of G with $\dim V_1 = n$ and $\dim V_2 = m$, and basis $\{v_1, \dots, v_n\}$, $\{w_1, \dots, w_m\}$

① $V_1 \oplus V_2$. vector space of dim. $n+m$

with basis $\{(v_1, 0), (v_2, 0), \dots, (0, w_1), (0, w_2) \cdots\}$

rep on $V_1 \oplus V_2$: $g\cdot(v, w) := (g\cdot v, g\cdot w)$ — G-action

$[(T_1 \oplus T_2)(g)](v \oplus w) := T_1(g)v \oplus T_2(g)w$. — rep.

mat. rep.

$$M_{T_1\oplus T_2}(g) = \begin{array}{c} \begin{array}{cc} n & m \end{array} \\ \left(\begin{array}{c|c} M_{T_1}(g) & 0 \\ \hline 0 & M_{T_2}(g) \end{array}\right) \end{array}$$

② $V_1 \otimes V_2$: vector space of dim $\underline{n\cdot m}$. basis

$\{v_i \otimes w_j : 1 \le i \le n.\ 1 \le j \le m.\}$

$$(\sum_i a_i v_i) \otimes (\sum_j b_j w_j) = \sum_{ij} a_i b_j \, v_i \otimes w_j$$

rep on $V_1 \otimes V_2$:

$$g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w)$$

$$[(T_1 \otimes T_2)(g)] (v \otimes w) := T_1(g) v \otimes T_2(g) w$$

$$[(M_1 \otimes M_2)(g)]_{ia, jb} = (M_1(g))_{ij} \, (M_2(g))_{ab}$$

③ The dual vector space. $V^\vee$ (or V^*)

$\{$ linear maps: $V \to k \} := \mathrm{Hom}(V, k)$

with $v_i^\vee$. $\quad v_i^\vee (v_j) = \delta_{ij}$

$\dim V^\vee = \dim V = n$.

(induced action on function space)

rep on $V^\vee$: $\quad (g \cdot v_i^\vee)(v_j) = v_i^\vee (g^{-1} \cdot v_j)$

natural pairing:

$$(g \cdot v_i^*) (g \cdot v_j) = v_i^* (g^{-1} \cdot g \cdot v_j)$$
$$= v_i^* (v_j) = \delta_{ij}$$

$$T(g) : V \to V \quad , \quad v \mapsto T(g) v$$

$$T^\vee(g) : V^\vee \to V^\vee \quad , \quad v^\vee \mapsto T^\vee(g) v^\vee$$

$$\underline{v_j = \sum_i M_{ij} v_i}$$

$$\underline{v_i^\vee(v_j)} = \sum_k M^\vee(g)_{ki} v_k^\vee \cdot \left(\sum_l M(g)_{lj} v_l \right)$$

$$= \sum_{kl} M^\vee(g)_{ki} M(g)_{lj} \, \underbrace{v_k^\vee(v_l)}_{\delta_{kl}}$$

$$= \underline{\sum_l M^\vee(g)_{li} M(g)_{lj} = \delta_{ij}}$$

$$\Leftrightarrow \quad M^{\vee}(g) = [M(g^{-1})]^{tr} = M(g)^{tr.-1}$$

8.4 Characters

For any finite-dimensional representation

$$T : G \to \mathrm{Aut}(V)$$

of any group G. we can define the character of the representation χ_T

$$\chi_T : G \to K$$

$$\chi_T(g) := \mathrm{Tr}_V(T(g))$$

1. equivalent $\Leftrightarrow$ same character function

$$\chi_T(h^{-1}gh) = \chi_T(g) \qquad \text{"class function"}$$

2. independent of basis choices

3. For above representations.

a. $M_{T_1 \oplus T_2}(g) = \begin{pmatrix} M_{T_1}(g) & 0 \\ 0 & M_{T_2}(g) \end{pmatrix}$

$$\chi_{T_1 \oplus T_2} = \chi_{T_1} + \chi_{T_2}$$

b. $(M_1 \otimes M_2)(g)_{ia,jb} = (M_1(g))_{ij} \, (M_2(g))_{ab}$

$$M^1 \otimes M^2 = \begin{pmatrix} m^1_{11} M^2 & & \\ & m^1_{22} M^2 & \\ & & \ddots \end{pmatrix} = \begin{pmatrix} m^1_{11} m^2_{11} & & \\ & m^1_{11} m^2_{22} & \\ & & \ddots \end{pmatrix}$$

$$= \sum_i m^1_{ii} \cdot \sum_j m^2_{jj}$$

$$\chi_{T_1 \otimes T_2} = \chi_{T_1} \cdot \chi_{T_2}$$

8.5 Haar measure (aka invariant integration)

Consider a function $f : G \to \mathbb{C}$. $f \in \mathrm{Map}(G, \mathbb{C})$

$$\langle f \rangle := \frac{1}{|G|} \sum_{g \in G} f(g) \quad \Longrightarrow \quad \text{"}\underline{\int_G dg}\, f(g)\text{"}$$

$$\int_G dg \in \left(\mathrm{Map}(G, \mathbb{C})\right)^\vee = \mathrm{Hom}(\mathrm{Map}(G, \mathbb{C}), \mathbb{C})$$

$$\int_G dg : f \mapsto \langle f \rangle$$

For finite group. $\underline{\frac{1}{|G|} \sum_{g \in G} f(hg) = \frac{1}{|G|} \sum_{g \in G} f(g)}$

invariant under left translation $L_h : g \mapsto hg$

We require similarly for $\int_G dg$.

$$\underline{\int_G f(hg)\, dg = \int_G f(g)\, dg} \qquad (\forall h \in G)$$

left invariance condition.

Left Haar measure.

(right Haar measure: $\int_G f(gh)\,dg = \int_G f(g)\,dg$)

1. For a finite group, left and right invariant measures are unique up to an overall scale.

holds also for compact Lie groups.

in general physics context: subset of $\mathbb{C}^m$.

compact $\Leftrightarrow$ closed & bounded

$$U(n) = \{ A \in GL(n,\mathbb{C}) \mid A^{+}A = \mathbb{1} \} \subset \mathbb{C}^{n^2}$$

$$\sum_j (A^{+})_{ij} A_{ji} = 1$$

$$\Rightarrow \sum_j |A_{ji}|^2 = 1 \Rightarrow |A_{ji}| \leq 1 \quad \forall ij$$

other examples: $Sp(n) \cong U(2n) \cap Sp(2n,\mathbb{C})$

$Sp(1) \cong SU(2)$

non-compact: $O(1,d)$

$Sp(2n,\mathbb{R}) \rightarrow \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \quad B^T = B$

$GL(n,\mathbb{R})$

2. Locall compact & Hausdorff

There exists a left invariant measure

on G. which is unique up to scale

(similar for right - invariance)

But $\underline{\text{Left} \neq \text{right.}}$

Examples

1. $G = \mathbb{R}$

$$\int_G dg\, f(g) = \int_G dg\, f(g+a) \qquad (a \in \mathbb{R})$$

$$\Rightarrow c\int_{-\infty}^{+\infty} dx\, f(x)$$

$$c\int_{-\infty}^{\infty} dx\, f(x+a) = c\int_{-\infty}^{\infty} d(x+a)\, f(x+a) = c\int_{-\infty}^{\infty} dx\, f(x)$$

2. $G = \mathbb{Z}$

$$\int_G dg\, f(g) = c\sum_{n\in\mathbb{Z}} f(n)$$

3. $G = \mathbb{R}^*_{>0}$

$$\int_G f(g)\, dg = c\int_0^\infty f(x)\frac{dx}{x}$$

$$\forall a \in \mathbb{R}^*_{>0}: \underline{\int_0^\infty f(ax)\frac{dx}{x}} = \int_0^\infty f(x)\frac{d(x/a)}{x/a} = \underline{\int_0^\infty f(x)\frac{dx}{x}}$$

4. $G = GL(n,\mathbb{R})$

$$\underline{g \longmapsto g_0 g = g'}$$

$$g'_{ij} = \sum_k (g_0)_{ik}\, g_{kj} \Rightarrow \frac{\partial g'_{ij}}{\partial g_{kl}} = (g_0)_{ik}\,\delta_{jl}$$

$$\prod_{ij} dg'_{ij} \longmapsto \underline{\left| \frac{\partial(g'_{11} \cdots g'_{nn})}{\partial(g_{11} \cdots g_{nn})} \right|} \prod_{ij} dg_{ij}$$

$$= |\det g_0|^n \prod_{ij} dg_{ij}$$

We thus define Haar measure

$$\int_{G=GL(n,\mathbb{R})} f(g)\, dg := c \int f(g) |\det g|^{-n} \prod_{ij} dg_{ij}$$

$$\int f(g_0 g) |\det g|^{-n} \prod dg_{ij}$$

$$= \int f(g) |\det g_0^{-1} g|^{-n} \prod \underline{d(g_0^{-1} g)_{ij}}$$

$$= \int f(g) |\det g|^{-n} \cancel{|\det g_0|^n} \cancel{|\det g_0|^{-n}} \prod dg_{ij}$$

$$= \int f(g) |\det g|^{-n} \prod dg_{ij}$$

5. $G = U(1) = \{ z \in \mathbb{C} .\ |z| = 1 \}$

$$z = e^{i\phi} \qquad dz = iz\, d\phi$$

$$\int_G f(g)\, dg = \frac{1}{2\pi i} \oint_{|z|=1} f(z) \frac{dz}{z} = \int_0^{2\pi} \frac{d\phi}{2\pi} g(\phi)$$

$$\left(\int_0^{2\pi} \frac{d\phi}{2\pi} = 1 \right)$$

6. $G = SU(2) \qquad g \in SU(2)$

$$g = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \qquad |\alpha|^2 + |\beta|^2 = 1$$

$$\alpha = e^{\frac{i}{2}(\phi + \psi)} \cos\frac{\theta}{2} \qquad \beta = i e^{i\frac{1}{2}(\phi - \psi)} \sin\frac{\theta}{2}$$

$\phi \in [0, 2\pi)$, $\theta [0, \pi)$. $\varphi \in [0, 4\pi)$

$$d\alpha\, d\bar{\alpha}\, d\beta\, d\bar{\beta} \longrightarrow \underbrace{\left| \frac{\partial(\alpha, \bar{\alpha}, \beta, \bar{\beta})}{\partial(r, \varphi, \phi, \theta)} \right|}_{J} dr\, d\varphi\, d\phi\, d\theta$$

$g \longmapsto g_0 g$, $|\det g_0| = 1$

$$J \big|_{r=1} \propto \sin\theta$$

Haar measure $\frac{1}{16\pi^2} \int dr\, d\varphi\, d\phi\, \sin\theta\, d\theta$

normalization

7. L & R Haar measure.

$$G = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \;\middle|\; x, y \in \mathbb{R},\ x > 0 \right\}$$

$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{x} & -\frac{y}{x} \\ 0 & 1 \end{pmatrix} \in G$$

$$\underbrace{\overset{g}{\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}}}_{g_0} \underbrace{\overset{g_0}{\begin{pmatrix} u & v \\ 0 & 1 \end{pmatrix}}}_{g} = \begin{pmatrix} x\underline{u} & xv + y \\ 0 & 1 \end{pmatrix} \in G$$

① left: $g \longmapsto g_0 g$

$$du\,dv \longmapsto x^2\, du\,dv$$

Haar measure: $\underline{\int x^{-2} dx\,dy}$

② right: $g \longmapsto g g_0$

$$dx\,dy \longmapsto u\,dx\,dy$$

Haar measure: $\cancel{\int u^{-1} du\,dv}$

$$\underline{\int x^{-1} dx\,dy}$$

<u>Proposition</u> If (T, V) is rep of a <u>compact group</u> G. and V is an inner product space

$\Rightarrow$ (T, V) is unitarizable.

If T is not already unitary w.r.t in product $\langle \cdot \rangle_1$. then we can define a new inner product

$$\langle v, w \rangle_2 := \int_G \underline{\underline{\langle T(g) v, T(g) w \rangle_1}} \, dg$$

$$\left(\hookrightarrow H = \sum_g T^+ T \right)$$

Then

$$\langle T(g) v, T(g) w \rangle_2 = \langle v, w \rangle_2$$