6. group action.

orbits., fixed points, stabilizer

Theorem. (Stab − orbit)

$$O_G(x) \xrightarrow{\cong} G/G^x$$

$$|O_G(x)| = [G : G^x]$$

$SO(3)$ acts on S^2. $\mathrm{Orb}_{SO(3)} = S^2$

$\mathrm{Stab}_{SO(3)}(\hat{z}) \cong SO(2)$

$$S^2 \cong SO(3)/SO(2)$$

$SU(2)$ on $\mathbb{C}^2$. $S^3 \cong SU(2)$

7. G-action on itself.

① H a subgroup, right action on G.

$gH = \{gh.\ h \in H\}$ left-cosets

$|gH| = |H|$

+ Lagrange. Finite G.

$$|G|/|H| = [G : H]$$

② action by conjugation.

orbits / conjugacy class

$C(h) = \{ g h g^{-1} \quad g \in G \}$

Stab-orb. $|C(g)| = [G : C_G(g)]$

↳ $\mathrm{Stab}_G(g)$

centralizer

Finite G. $|C(g)| = \frac{|G|}{|C_G(g)|}$

$+ \quad |G| = \sum |C(g)|$

$\Rightarrow \quad |G| = \sum_{\{C(g)\}} \frac{|G|}{|C_G(g)|}$ "class equation"

↳ ① $|G| = p^n \Rightarrow Z(G) \neq \{e\}$

② (Cauchy) $p \mid |G|$

$\Rightarrow \exists g \in G.\ g^p = 1$

↳ class function.

function f on G.

$f(g g_0 g^{-1}) = f(g_0) \quad \forall g_0, g \in G.$

↳ meet rep.

$\chi_T(g) = \mathrm{Tr}\, T(g)$ character

↳ equivalent rep. φ_1, φ_2 $\exists g_2$. s.t.

$\varphi_2(g) = g_2 \varphi_1(g) g_2^{-1} \quad \forall g_1 \in G.$

mat. rep. $T_1 : G \to GL(n.k)$

$T_2 \; G \to GL(n.k)$

$\exists S \in GL(n.k)$. s.t.

$T_2(g) = S T_1(g) S^{-1} \quad \forall g \in G$.

8. Morphisms of G spaces / equivariant map

$f : X \to X'$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\Phi(g)"'] & X' \arrow[d, "\Phi'(g)"] \\ X \arrow[r, "f"] & X' \end{tikzcd}$$

$f(\Phi(g, x)) = \phi'(g, f(x))$

$f(gx) = g \cdot f(x)$

9. The symmetric group S_n.

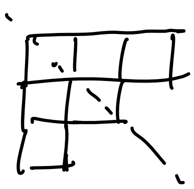
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} = (1243)$

① unique <u>cycle decomposition</u>. of $\phi \in S_n$

② r-cycles are conjugate

↳ conjugacy classes labeled by partitions of n.

S_6. $\vec{\lambda} = \{3. 2. 1\}$



Young diagram.

$\mathrm{sgn}: S_n \longrightarrow \mathbb{Z}_2$

$\phi \longmapsto \mathrm{sgn}(\phi) = (-1)^{n-t}$ → len. of cycle decomposition.

$A_n \lhd S_n$ $\qquad \mathrm{sgn}(\phi \in A_n) = 1$

Why S_n?

finite G. of order n.

$\xrightarrow{\text{embed}} S_n$

$\cong$ some subgroup of S_n

$D_4 \cong S_4 \subset S_8 \qquad (|D_4| = 8)$

10. quotient groups

$N \lhd G$. then G/N has a natural group structure

$(g_1 N)\cdot(g_2 N) := (g_1 g_2)\cdot N$.

$\mu: G \longrightarrow G/N$.

$g \longmapsto gN$

$\ker\mu = N$.

1st. isomorphism theorem $\quad \mu: G \to G'$

$G/\ker\mu \cong \mathrm{im}\,\mu$.

11. exact sequence.

$$\to G_{i-1} \xrightarrow{f_{i-1}} G_i \xrightarrow{f_i} G_{i+1}$$

$$\mathrm{im}\, f_{i-1} = \ker f_i$$

SES. $1 \xrightarrow[f_0]{} G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \xrightarrow[f_3]{} 1$

① $\ker f_1 = \mathrm{im}\, f_0 = \{1\}$ $\quad f_1$ injective

② $\mathrm{im}\, f_2 = \ker f_3 = G_3$ $\quad f_2$ surjective.

$$1 \to N \to G \to Q \to 1$$

G is an extension of Q by N.

$$\begin{cases} N \cong H \triangleleft G \\ Q \cong G/N \end{cases}$$

↳ Central extension: $N \subset Z(G)$

$$1 \to \mathbb{Z}_2 \to SU(2) \to SO(3) \to 1$$

8. Representation theory

8.0. Some motivation

1 In QM. symmetries are represented by unitary / linear, antiunitary / antilinear operators in Hilbert space $\mathcal{H}$.

(Wigner. 1931 ; Weinberg. QFT-I. 1995)

If the Hamiltonian H has certain symmetry. represented by U. $U^{+} H U = H$. / $[H, U] = 0$

They have the same eigen states.

$\Rightarrow$ simultaneous diagonalization.

$$H = t \sum_{\langle ij \rangle} C_i^{+} C_j + h.c.$$

$$H = \begin{pmatrix} 0 & t & & & & & \\ t & 0 & t & & & 0 & \\ & t & 0 & t & & & \\ & & t & 0 & t & & \\ & & & t & 0 & t & \\ & 0 & & & t & 0 & t \\ & & & & & t & 0 \end{pmatrix}$$

$$C_k^{+} = \sum_i e^{i k r_i} C_i^{+} \ / \ C_i^{+} = \sum_k e^{-i k r_i} C_k^{+}$$

$$\Rightarrow \tilde{H} = 2t \sum_{k_i} \cos k_i a \, C_k^{+} C_k$$

$$k_i = \frac{2\pi i}{aN} \quad i = 0, \cdots N-1$$

$$\tilde{H} = \begin{pmatrix} 2t\cos k_1 & & \\ & 2t\cos k_2 & \\ & & \ddots \end{pmatrix}$$

eigen space labeled by k_i

2 symmetry $\Leftrightarrow$ selection rules $\quad [H, W] = 0$

$\Rightarrow \exists S$. s.t

block-diagonal.

$$SHS^{-1} = \begin{pmatrix} H_1 & 0 & 0 \\ 0 & H_2 & 0 \\ 0 & 0 & H_3 \end{pmatrix}$$

symmetry sectors labeled by (a set of) different quantum numbers

e.g. for Fermions . QN. = $\begin{cases} \text{particle number} \\ S \\ S_z \end{cases}$

3. Conservation laws.

Noether's theorem:

Continuous symmetry $\Leftrightarrow$ classically conserved current.

Review of basic definitions

① $G \to GL(V)$

V some vector space over field K

$GL(V)$ / $Aut(V)$: invertible linear transformations $V \to V$.

② rep. of G is a group homomorphism.

$$T: G \to GL(V)$$
$$g \mapsto T(g)$$

(T, V) denotes the representation.

$$T(g_1) T(g_2) = T(g_1 g_2) \quad \forall g_1, g_2 \in G.$$

V is called the carrier space / representation space.

Given an ordered basis of finite dim V.

$$\{\hat{e}_1, \cdots \hat{e}_n\} \Rightarrow GL(V) \cong GL(n, K)$$

$$\underline{T(g)\hat{e}_i = \sum_j M(g)_{ji} \hat{e}_j}$$

$$\begin{aligned}
T(g_1)[T(g_2)\hat{e}_i] &= T(g_1) \sum_j M(g_2)_{ji} \hat{e}_j \\
&= \sum_j M(g_2)_{ji} (T(g_1)\hat{e}_j) \\
&= \sum_j M(g_2)_{ji} \sum_k M(g_1)_{kj} \hat{e}_k \\
&= \sum_k (M(g_1) M(g_2))_{ki} \hat{e}_k
\end{aligned}$$

$$T(g_1)T(g_2) = T(g_1 g_2) \Leftrightarrow M(g_1)M(g_2) = M(g_1 g_2)$$

In terms of group actions. rep. of G is a G-action on a vector space that respects linearity

$$g\cdot(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 g\cdot v_1 + \alpha_2 g\cdot v_2 \qquad v_i \in V, \ \alpha_i \in K$$

Examples.

1. rep. of degree / dim 1.

$$T : G \to \mathbb{C}^*$$

for element of order n. $g^n = 1_G$

$T(g)^n = 1$ $\quad$ $T(g)$ are roots of 1

$\mathbb{Z}_3 \cong \mu_3 \cong A_3 = \langle g \rangle$ $\quad$ $T(g) = \omega = e^{i\frac{2\pi}{3}} / e^{i\frac{4\pi}{3}}$

if take $T(g) = 1$ $\quad$ trivial representation (unit)

2. "regular representation" of a finite group. (more to be discussed later)

Let $\dim V = |G| = n$. with an ordered basis set $\{\hat{e}_g\}$ $(g \in G)$

$$T(g_1)\cdot \hat{e}_{g_2} = \hat{e}_{g_1 g_2}$$

V	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$\langle a, b \mid a^2 = b^2 = (ab)^2 = e \rangle$ ⑩

$\cong \mathbb{Z}_2 \times \mathbb{Z}_2$

$e = (0, 0)$

$a = (1, 0)$

$b = (0, 1)$

$c = (1, 1)$

$$T_{reg} : \mathbb{Z}_2 \times \mathbb{Z}_2 \to GL(V) \ (\dim V = 4)$$

$$V = \{\hat{e}_e, \hat{e}_a, \hat{e}_b, \hat{e}_c\}$$

$T(e)\hat{e}_g = \hat{e}_g$ $\qquad \underline{T(e) = \mathbb{1}_4}$ $\qquad \begin{cases} \chi(T(e)) = \dim V = 4 \\ \chi(T(g \neq e)) = 0 \end{cases}$

$$\begin{cases} T(a)\hat{e}_e = \hat{e}_a \\ T(a)\hat{e}_a = \hat{e}_e \\ T(a)\hat{e}_b = \hat{e}_c \\ T(a)\hat{e}_c = \hat{e}_b \end{cases}$$

$$T(a) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

3. more generally. G acts on set X

$$x \mapsto gx$$

Let V be a vector space with basis $\{e_x\}$ $(x \in X)$

$$T(g)e_x = e_{gx}$$

permutation representation.

4. $G = \mathbb{Z}, \mathbb{R}, \mathbb{C}$ $\qquad T : G \to GL(\mathbb{C})$

$$n \mapsto a^n \quad (a \in \mathbb{C}^*)$$

$$n_1 + n_2 \to a^{n_1} \cdot a^{n_2} = a^{n_1+n_2}$$

5. $G = \mathbb{Z}, \mathbb{R}, \mathbb{C}$. $\quad T : G \longrightarrow GL(2, k)$

$$n \longmapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

6. $G = GL(n, k)$ $\rightarrow$ one-dim. representation

$$T(g) := |\det g|^{\mu}$$

$$T(g_1 g_2) = |\det(g_1 g_2)|^{\mu} = |\det g_1|^{\mu} |\det g_2|^{\mu}$$
$$= T(g_1) T(g_2)$$

7. 1+1 dim Lorentz group

$$x^{0\prime} = \cosh\theta\, x^0 + \sinh\theta\, x^1$$

$$x^{1\prime} = \sinh\theta\, x^0 + \cosh\theta\, x^1$$

$$\begin{pmatrix} x^{0\prime} \\ x^{1\prime} \end{pmatrix} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix} = B(\theta) \begin{pmatrix} x^0 \\ x^1 \end{pmatrix}$$

($B(\theta) \in O(1,1) = \{ A \mid A^T \eta A = \eta \}$, $\eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$)

$$B(\theta_1) \cdot B(\theta_2) = B(\theta_1 + \theta_2)$$

<u>Definition</u>. Let (T_1, V_1) and (T_2, V_2) be two reps. of a group G. An intertwiner (intertwining map 交结映射) between these two reps is a linear transformation
$A: V_1 \to V_2$
s.t. $\forall g \in G$. the following diagram commutes.

$$\begin{tikzcd} V_1 \arrow[r, "A"] \arrow[d, "T_1(g)"'] & V_2 \arrow[d, "T_2(g)"] \\ V_1 \arrow[r, "A"] & V_2 \end{tikzcd}$$

i.e. $T_2(g) A = A \cdot T_1(g)$

A is an equivariant linear map of G spaces $V_1 \to V_2$

$\mathrm{Hom}_G(V_1, V_2)$: vector space of all intertwiners.

<u>Definition</u>. Two reps (T_1, V_1) and (T_2, V_2) are equivalent $(T_1, V_1) \cong (T_2, V_2)$ if there is an intertwiner $A: V_1 \to V_2$ which is an <u>isomorphism</u>, that is

$$T_2(g) = A T_1(g) A^{-1} \quad (\forall g \in G)$$

— Unitary representations

Let V be a complex vector space over $\mathbb{C}$.

Define the inner product on V as a sesquilinear map. $\langle \cdot , \cdot \rangle : V \times V \longrightarrow \mathbb{C}$. obeying

(1) $\langle v, * \rangle$ is linear for all fixed v.

(2) $\langle w, v \rangle = \overline{\langle v, w \rangle}$

(3) $\langle v, v \rangle \geq 0$. $= 0$ iff $v = 0$

sesquilinear:

$$\left(\begin{aligned} &\langle v, \alpha_1 w + \alpha_2 w_2 \rangle = \alpha_1 \langle v_1, w_1 \rangle + \alpha_2 \langle v_1, w_2 \rangle \\ &\langle \alpha_1 v_1 + \alpha_2 v_2, w \rangle = \bar{\alpha}_1 \langle v_1, w \rangle + \bar{\alpha}_2 \langle v_2, w \rangle \end{aligned} \right)$$

Definition. Let V be an inner product space

A unitary rep is a rep (V, U) s.t. $\forall g \in G$ $U(g)$ is a unitary operator on V. i.e.

$$\langle U(g) v, U(g) w \rangle = \langle v, w \rangle \quad \forall v, w \in V \quad \forall g \in G.$$

Definition. If a rep (V, T) is equivalent to a unitary rep. then it is said to be unitarizable.