$$\begin{cases} [(QP)\Psi](\omega^k) = [Q\cdot(P\Psi)](\omega^k) = \omega^k (P\Psi)(\omega^k) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad = \omega^k \Psi(\omega^{k-1}) \\ [P(Q\Psi)](\omega^k) \end{cases}$$

$$\phi : G \times X \longrightarrow X$$
$$(g, x) \longmapsto g\cdot x$$

$\Rightarrow$ Induced action on the vector space of functions on X : $\mathcal{F}[X \to K]$

$$\begin{cases} f_1, f_2 \in \mathcal{F} \qquad f_1(x) = a \in k \quad f_2(x) = b \in k \\ (f_1 + f_2)(x) = f_1(x) + f_2(x) \\ c\cdot f \in \mathcal{F} \end{cases}$$

$$\tilde{\phi}(g, f)(x) = f(\phi(g^{-1}, x))$$
$$(g\cdot f)(x) = f(g^{-1}x)$$
$$g_1\cdot g_2 f(x) = g_2 f(g_1^{-1}x) = f(g_2^{-1}g_1^{-1}x) = (g_1 g_2)\cdot f(x)$$

Application: stabilizer code.

$P^n = (P^1)^{\otimes n} \qquad P^1 = \{\pm I, \pm i, \pm X, \pm Y, \ldots\}$

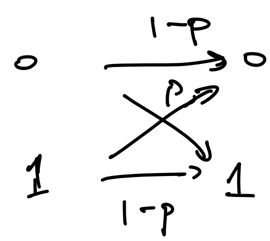
$X|0\rangle = |1\rangle \qquad Z|0\rangle = |0\rangle$

$X|1\rangle = |0\rangle \qquad Z|1\rangle = -|1\rangle$

Select a subgroup $S \subset P^n$

$V_S = \{|\psi\rangle : s|\psi\rangle = |\psi\rangle, \ \forall s \in S\} \subset \mathcal{H}^n$

Quantum error

① 

$(p \ll 1)$

$\alpha|0\rangle + \beta|1\rangle \xrightarrow{\text{noise}} \alpha|1\rangle + \beta|0\rangle$

② $|0\rangle \longrightarrow |000\rangle$

$|1\rangle \longrightarrow |111\rangle$

$\alpha|001\rangle + \beta|110\rangle \xrightarrow{?} \alpha|000\rangle + \beta|111\rangle$

$\downarrow X_3$

$\alpha|000\rangle + \beta|111\rangle$

Stabilizer formalism (3-qubit)

$S = \{I, Z_1Z_2, Z_2Z_3, Z_1Z_3\} = \langle Z_1Z_2, Z_2Z_3\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

$Z|0\rangle = |0\rangle$

$Z|1\rangle = -|1\rangle$

Z_1Z_2: $\underline{|000\rangle}$, $|001\rangle$, $|110\rangle$, $\underline{|111\rangle}$

Z_2Z_3: $\underline{|000\rangle}$, $|100\rangle$, $|011\rangle$, $\underline{|111\rangle}$

$V_S = \text{span}\{|000\rangle, |111\rangle\}$

Error set : $\langle X_1, X_2, X_3 \rangle$

$\{X_1, Z_1\} = 0$

If E anticommutes with $s \in S$

$s|\varphi\rangle = |\varphi\rangle$

$s\underline{E|\varphi\rangle} = -Es|\varphi\rangle = -|\varphi\rangle \qquad E|\varphi\rangle \in \underline{V_S^\perp}$

$\Rightarrow$ detectable

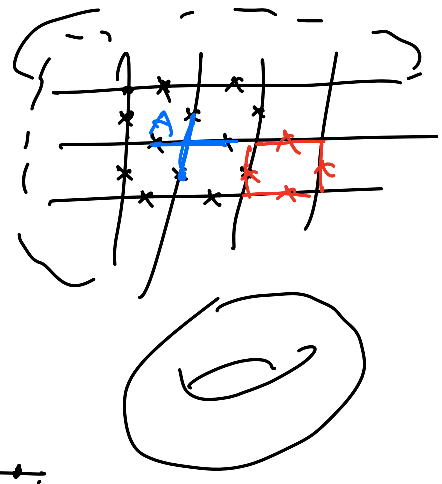
If E commutes $\forall s \in S \qquad (E \in N(S) - S)$

$N(S) = \{ g \in P^n,\ gs = sg,\ \forall s \in S \}$

$s\underline{E|\varphi\rangle} = Es|\varphi\rangle = \underline{E|\varphi\rangle} \qquad E|\varphi\rangle \in \underline{V_S}$

$\Rightarrow$ undetectable

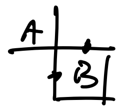
Toric code (Kitaev , Ann. Phys. 2006)



$$A_v = \prod_{j \in \text{star } v} Z_j$$

$$B_p = \prod_{j \in \text{plaq.}} X_j$$

$$H = -\sum_v A_v - \sum_p B_p$$



$$[A_v, B_p] = 0$$

$S = \langle \{A_v\}, \{B_p\} \rangle$ stabilize the code space V_S

N u.c. $\quad 2^{2N}$ $\qquad A|\varphi\rangle = |\varphi\rangle$ $\qquad A_v^2 = B_p^2 = 1$

$B_p|\varphi\rangle = |\varphi\rangle$

every A / B cuts the space in half

$2N$ operators , + $\prod A = \prod B = 1$ (only $N-1$ A/B independent)

$\hookrightarrow$ $2(N-1)$ constraints

$$2^{2N-(2N-2)} = 2^2 = 4$$

l-bit, k-independent generators of S

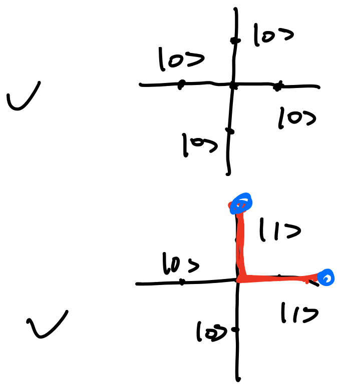
$$\dim C(S) = 2^{l-k}$$

$l = 2N$

$k = 2N-2$

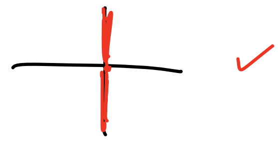
$\Rightarrow$ Toric code encodes two qubits.

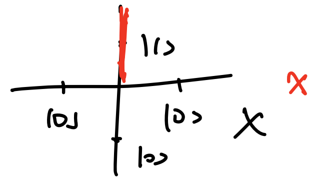
A :



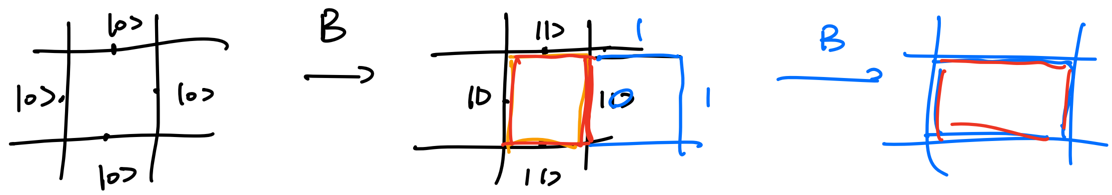
$Z|0\rangle = |0\rangle$

$Z|1\rangle = -|1\rangle$



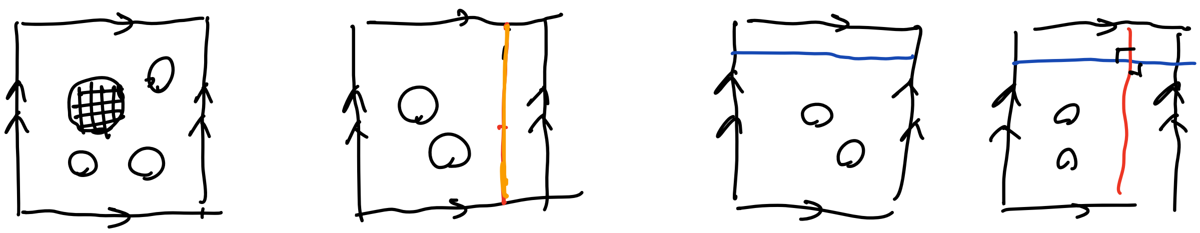


$B = \prod X$



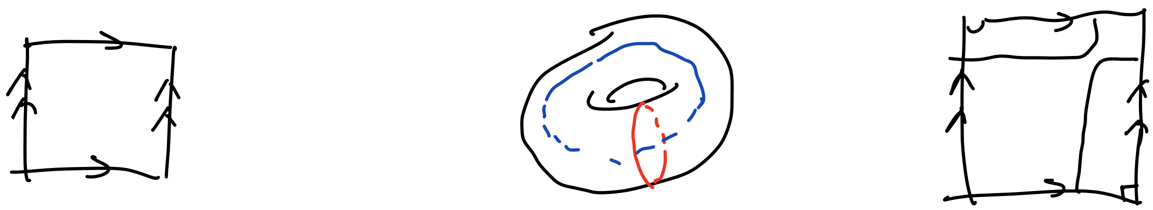
$\Longrightarrow$ GS = equal weight superposition of all closed loops

typical config.



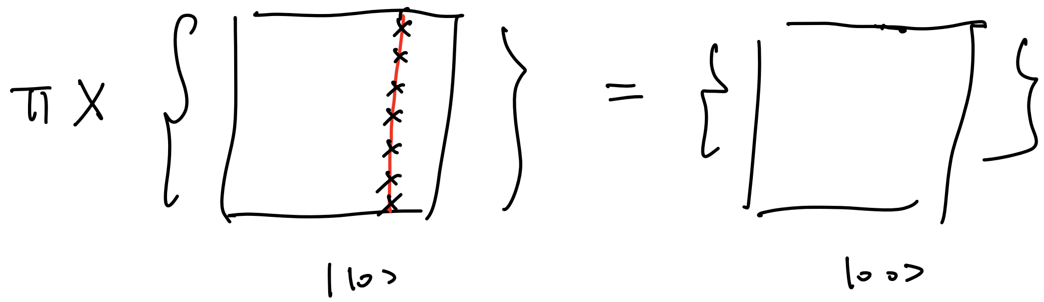
$Z_2 \times Z_2$ (0, 0) (1, 0) (0, 1) (1, 1)

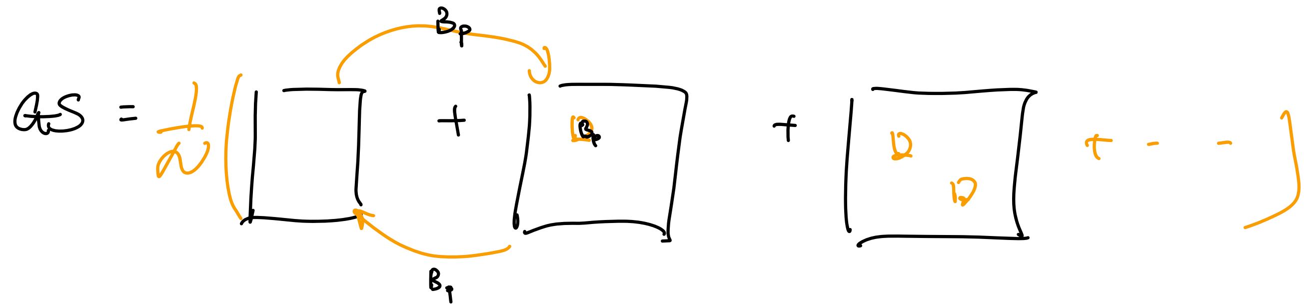
$\Downarrow$ $|00\rangle$ $|10\rangle$ $|01\rangle$ $\Downarrow$ $|11\rangle$



local noise / error $\prod Z$, $\prod X$, suppressed.

Bit operations via string operators across the lattice


ΠX
=
|10⟩
|00⟩


GS = 1/N
B_p
+
+ - - -

Review of Group part

1. Definition of groups $(G, e, \underline{m}, \underline{I})$

① set G

② $e \in G$ $\quad e \cdot g = g \cdot e = g$.

③ $m: G \times G \to G$

④ $I: G \to G$

$G = \mathbb{Z}, \mathbb{R}, \mathbb{C}$ groups if $m =$ "+"

not if $m =$ "×"

$\mathbb{R}^* = \mathbb{R} - \{0\}$, $\mathbb{C}^* = \mathbb{C} - \{0\}$

↪ $|G|$ order $\begin{cases} \text{finite} \\ \text{infinite} \end{cases}$ group

↪ $g_1 g_2 = g_2 \cdot g_1 \quad \forall g_1, g_2 \to$ abelian

↳ noabelian.

2. Direct product $H \times G$.

$(h_1, g_1) \cdot (h_2, g_2) = (h_1 \cdot h_2, g_1 \cdot g_2)$

↳ semidirect product $H \rtimes_\alpha G$. $h \in H$. $g \in G$.

$$(\underline{h_1}, \underline{g_1}) \cdot_\alpha (\underline{h_2}, \underline{g_2}) = (\underline{h\,\alpha_{g_1}(h_2)}, \underline{g_1 g_2})$$

$$\{R_1|\tau_1\}\{R_2|\tau_2\}\vec{r} = \{R_1|\tau_1\}(R_2\vec{r} + \vec{\tau}_2)$$
$$= R_1R_2\vec{r} + R_1\vec{\tau}_2 + \vec{\tau}_1$$

$$(\tau_1, R_1)(\tau_2, R_2) = (R_1\vec{\tau}_2 + \vec{\tau}_1, R_1R_2)$$

↳ symmorphic space groups

3. subgroups $H \subset G$.

$\underline{m}$: $H \times H \to H$

$\underline{I}$: $H \to H$

G has trivial subgroups $\{e\}$ and G.

proper subgroup $H \neq G$

$\mathbb{Z} \subset \mathbb{R} \subset \mathbb{C}$ "+"

↳ $H \triangleleft G$: $gHg^{-1} = H$ $(\forall g \in G)$

↳ simple group, no nontrivial normal subgroup.

↳ centralizer $C_G(h) = \{g \in G : gh = hg\} \subset G$

$C_G(H) = \{g \in G . gh = hg, \forall h \in H\} \subset G$

normalizer $N_G(H) = \{g \in G . gHg^{-1} = H\}$

$$C_G(H) \subset N_G(H)$$

4. $GL(n, k)$

$\hookrightarrow SL(n, k)$

$$\left.\begin{matrix} O(n, k), & SO(n, k) \\ U(n, k), & SU(n, k) \end{matrix}\right\} \underline{\det}$$

$$A^T \underline{J} A = \underline{J} \begin{cases} J_{p,q} = \begin{pmatrix} -1_p & \\ & 1_q \end{pmatrix} \\ \text{symplectic} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{cases}$$

Homomorphism / isomorphism.

5. homomorphism. $\varphi : G \longrightarrow G'$

$$\begin{tikzcd} G \times G \arrow[r, "m"] \arrow[d, "\varphi \times \varphi"'] & G \arrow[d, "\varphi"] \\ G' \times G' \arrow[r] & G' \end{tikzcd}$$

$$\varphi(g_1) \underset{G'}{\cdot} \varphi(g_2) = \varphi(g_1 \underset{G}{\cdot} g_2)$$

$$\begin{cases} \varphi(e) = e' \\ \varphi(g^{-1}) = \varphi(g)^{-1} \end{cases}$$

ker / im : $\ker \varphi = \{ g \in G : \varphi(g) = 1_{G'} \}$

$\operatorname{im} \varphi = \varphi(G)$

① $\pi: SU(2) \longrightarrow SO(3)$

$$u\, \vec{x}\cdot\vec{\sigma}\cdot u^{+} := (\pi(u)\cdot\vec{x})\cdot\vec{\sigma}$$

$$\ker \pi = \{\pm \mathbb{1}\} \cong \mathbb{Z}_2$$

② $\Gamma: G \rightarrow \underline{GL(V)}$ $\quad V$ some n-dim vector space over k

given basis

$$GL(V) \cong \underline{GL(n.k)}$$

<u>isomorphism</u> : homo. + (1-1 & onto)

1-1: $\ker\varphi = \{e\}$

onto: $\varphi(G) = G'$

$\varphi: G \to G$: $Aut(G)$

isomorphism defines an equivalence relation

$$\mu_N \cong \mathbb{Z}_N$$

<u>matrix-rep.</u> $\quad T: G \rightarrow GL(n.k)$

$$T(g)\hat{e}_i = T(g)_{ij}\,\hat{e}_j$$

↳ equivalent rep $T \cong T'$ $\quad \exists S$ s.t.

$$T'(g) = S\,T(g)\,S^{-1} \quad \forall g \in G.$$

more generally <u>conj. rep</u> $\quad \varphi_{1,2}: G \rightarrow G'$

$$\varphi_2(g_1) = g_2\,\varphi_1(g_1)\,g_2^{-1}$$

6. define group action by homomorphism.

$$\Phi : G \to S_X := \{ X \xrightarrow{f} X \}$$

set of permutations

$$g \mapsto \phi(g, \cdot)$$

$$\Phi_g(x) = \phi(g, x) = g \cdot x$$

$$g_1 \cdot (g_2 \cdot x) = (g_1 \cdot g_2) x$$

↳ orbits. $Orb_G(x) = \{ g \cdot x . g \in G \}$

① defines equivalence relation

$x \sim y . \quad y = g \cdot x$

② orbits partition G.

$$O_G(x) = O_G(x') \text{ or}$$

$$O_G(x) \cap O_G(x') = \emptyset .$$

X/G set of orbits

↳ fixed points

$$Fix_X(g) = \{ x \in X : gx = x \} \subset X$$

→ stabilizer.

$$Stab_G(x) := \{ g \in G . g \cdot x = x \} \subset G .$$

(G^x)

a <u>group action</u> is

1. effective : $Fix_X(g) \neq X \quad \forall g \neq e$

2. transitive : $Orb_G(x) = X \quad \forall x \in X$

3. free : $Fix_X(g) = \phi \quad \forall g \neq e$

Theorem (Stabilizer - orbit)

$$O_G(x) \xrightarrow{\cong} G/G^x$$
$$g \cdot x \longmapsto gG^x$$

$$|O_G(x)| = [G : G^x]$$