Recap.

① normal subgroup $\quad gNg^{-1} = N \quad \forall g \in G \Rightarrow N \triangleleft G$

- D_4

② Simple group: no nontrivial normal subgroup

$\{1\}, \ G \triangleleft G.$

a. $\mathbb{Z}_p$, p prime

b. $A_n \ (n \geqslant 5)$

③ $\cdots G_{i-1} \xrightarrow{f_{i-1}} G_i \xrightarrow{f_i} G_{i+1} \xrightarrow{f_{i+1}} \cdots$

$\text{im } f_{i-1} = \ker f_i$: exact sequence

正合序列

$1 \xrightarrow{f_0} G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \xrightarrow{f_3} 1 \quad$ SES

① exactness on G_1: $\quad \text{im } f_0 = \{1_{G_1}\}$

$= \ker f_1$

$\Rightarrow f_1$ injective

② G_2: $\ker f_2 = \text{im } f_1$

③ G_3: $\text{im } f_2 = \ker f_3 = G_3 \Rightarrow f_2$ surjective

$\mu : G \to G'$

$$1 \to K \equiv \ker\mu \to G \xrightarrow{\mu} \operatorname{im}\mu \to 1$$

$G/K \cong \operatorname{im}\mu$ (1st isomorphism theorem)

$1 \to N \to G \to Q \to 1$ a SES

$G/N \cong Q$

G is an extension of Q by N.

Example. $1 \to G_1 \to G_1 \times G_2 \xrightarrow{\mu} G_2 \to 1$

$\quad\quad\quad\quad\quad\quad G_2 \quad\quad\quad\quad\quad\quad\quad\quad\quad G_1$

$(g_1, g_2) \longmapsto g_2$

$\quad\quad\quad\quad\quad\quad g_1$

$\varphi : \mu_{n^2} \longrightarrow \mu_n$

$z \longmapsto z^n$

$$1 \to \mathbb{Z}_N \to \mathbb{Z}_{N^2} \to \mathbb{Z}_N \to 1$$

Examples (cont.)

1. det : $O(n) \longrightarrow \mathbb{Z}_2$ $\qquad AA^T = \mathbb{1} \Rightarrow \det = \pm 1$

$$M \longmapsto \det(M)$$

$$\ker(\det) = SO(n)$$

$$1 \to SO(n) \to O(n) \to \mathbb{Z}_2 \to 1$$

1.b. det $U(n) \to U(1)$

$$1 \to SU(n) \to U(n) \to U(1) \to 1$$

2. $\pi : SU(2) \longrightarrow SO(3)$

$$\underline{u}\, \vec{x}\cdot\vec{\sigma}\, \underline{u^{-1}} = (\pi(u)\cdot\vec{x})\cdot\vec{\sigma}$$

$\pi(u) = \pi(-u)$

① $\ker \pi \cong \mathbb{Z}_2 = \{\pm \mathbb{1}_2\}$

② π is surjective

$$1 \to \mathbb{Z}_2 \to \underline{SU(2)} \to \underline{SO(3)} \to 1$$

$$SO(3) \cong SU(2)/\mathbb{Z}_2 \sim S^3/\mathbb{Z}_2 \cong \mathbb{R}P^3$$

recall : $R : SU(2) \times SU(2) \longrightarrow SO(4)$

$$\ker(R) \cong \mathbb{Z}_2$$

more generally:

$$1 \to \mathbb{Z}_2 \to \underline{Spin(n)} \to SO(n) \to 1$$

$$Spin(3) \cong SU(2)$$

$$Spin(4) \cong SU(2) \times SU(2)$$

$$\vdots$$

$$\mathbb{Z}_2 = \{\pm \mathbb{1}_2\} = Z(SU(2))$$

<u>Definition</u> (central extension)

$$1 \to A \xrightarrow{i} E \xrightarrow{\pi} G \to 1$$

1. A is abelian.
2. $A \subset Z(E)$ $\qquad i(a)\, b = b\, i(a) \quad (\forall a \in A, \forall b \in E)$

$$1 \to N \to G^{\text{Quantum}} \to G^{\text{classical}} \to 1$$

<u>Motivation for such extensions.</u> in QM

Physical states are "rays" in Hilbert space.

ray: $\dfrac{|\varphi\rangle}{\langle\varphi|\varphi\rangle}$ normalized vectors

$|\varphi\rangle, |\varphi'\rangle \in R \qquad |\varphi'\rangle = z\,|\varphi\rangle \qquad |z| \in U(1)$

A symmetry operation $\overset{T}{\cup}$ preserves probability.

$$|\langle\phi|\varphi\rangle|^2 = |\langle\phi'|\varphi'\rangle|^2 \quad \text{if} \quad \phi \in R_1 \;\; \varphi \in R_2$$
$$\phi' \in R_1' \;\; \varphi \in R_2'$$

$$R_i \xrightarrow{T} R_i'$$

(Wigner's theorem) symmetry operators U are unitary / linear or antiunitary / antilinear

operations on rays: $R \xrightarrow{T_1} R_1 \xrightarrow{T_2} R_2$ $\quad (T_2 \cdot T_1) = (T_2 T_1)$

$$R \xrightarrow{T_2 T_1} R_1$$

$U(T)$ representation on Hilbert space

$$U(T_2)\, U(T_1)\, \varphi_n = e^{i\alpha_n (T_2, T_1)}\, U(T_2 T_1)\, \varphi_n$$

$$U(T_2)\, U(T_1) = \underline{\underline{e^{i\alpha(T_2, T_1)}}}\, U(T_2 T_1)$$

"projective representation":

$$\theta_1 + \theta_2 = 2\pi \quad \text{around } \hat{z}.$$

SO(3)

classical: $R(\theta_1)\, R(\theta_2) = R(\theta_1 + \theta_2) = 1$

spin-1/2: $\quad R(\theta_1)\, R(\theta_2) = \underline{(-1)}\, R(\theta_1 + \theta_2)$

rep. of SU_2 : $\underline{U = \exp(i \sum \theta_i \sigma_i / 2)}$

$$U(\theta_1) U(\theta_2) = U(\theta_1 + \theta_2)$$

3. finite Heisenberg group.

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad Q = \begin{pmatrix} \omega & & & 0 \\ & \omega^2 & & \\ & & \omega^2 & \\ 0 & & & 1 \end{pmatrix}$$

$$\omega = e^{i\frac{2\pi}{4}}$$

$$\underline{\underline{QP = \omega PQ}} \quad \leftarrow$$ Weyl relation of canonical commutation relation.

Some background.

$[q, p] = i\hbar \quad (\hbar = 1)$

$qp - pq = i \qquad p, q$. acts on $f(\mathbb{R})$

$\Rightarrow \quad A = e^{i\xi p} \qquad B = e^{i\eta q} \qquad$ (Weyl.)

$AB = e^{i\xi p} \cdot e^{i\eta q} \qquad e^X e^Y = e^Z$

$$= e^{i(\xi p + \eta q) + \frac{1}{2}[i\xi p, i\eta q]}$$

$$\underline{Z = X + Y + \frac{1}{2}[X, Y]} + \frac{1}{12}\left([X, [X, Y]] - [Y, [X, Y]]\right) + \cdots$$

$$BA = e^{i(\xi p + \eta q) + \frac{1}{2}[i\eta q, i\xi p]}$$

$\Rightarrow \quad AB = e^{i\xi\eta} BA \equiv \omega BA \qquad (A, B : n \times n$ mats.)

$\det(AB) = \omega^n \det(BA) \Rightarrow \underline{\omega^n = 1}$

$$\begin{cases} A^k B = \omega^k B A^k \\ A B^l = \omega^l B^l A \end{cases} \Rightarrow A^k B^l = \omega^{kl} B^l A^k$$

$$k = n \,.\, l = 1 \Rightarrow \underline{A^n B} = \overset{1}{\cancel{\omega^n}} B A^n \overset{?}{\Rightarrow} A^n = \mathbb{1}$$

similary $B^n = \mathbb{1}$.

A general element in $\underline{\text{Heis}_N}$ has the form

$$\omega^a P^b Q^c$$

$$(\omega^{a_1} P^{b_1} Q^{c_1}) \cdot (\omega^{a_2} P^{b_2} Q^{c_2}) = \omega^{a_3} P^{b_3} Q^{c_3}$$

$$\begin{cases} a_3 = a_1 + a_2 + c_1 b_2 \\ b_3 = b_1 + b_2 \\ c_3 = c_1 + c_2 \end{cases}$$

$$\pi : \text{Heis}_N \longrightarrow \mathbb{Z}_N \times \mathbb{Z}_N$$

$$\omega^a P^b Q^c \longmapsto (b \bmod N,\ c \bmod N)$$

$$\ker(\pi) = \{ \omega^a \overset{\mathbb{1}}{\cancel{P^{rN}}} \overset{\mathbb{1}}{\cancel{Q^{sN}}} \} \cong \mathbb{Z}$$

$$1 \longrightarrow \mathbb{Z} \longrightarrow \text{Heis}_N \longrightarrow \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow 1$$

7. Group actions (cont.)

Recall that the group action of G on a set X:

$$\phi : G \times X \to X$$

1. left action. $\phi(g_1, \phi(g_2, x)) = \phi(g_1 g_2, x)$

$$g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$$

(right action $(x \cdot g_2) \cdot g_1 = x \cdot (g_2 g_1)$)

2. $\phi(1_G, x) = x$

A G-action is:

1. effective: $\forall g \neq 1,\ \exists x$. s.t. $gx \neq x$

   (ineffective $\exists g \neq 1,\ \forall x$. s.t. $gx = x$)

2. transitive: $\forall x, y \in X$. $\exists g$. s.t. $\underline{y = g \cdot x}$

   there is only one orbit

3. free: $\underline{\forall g \neq 1}$. $\underline{\forall x}$. $\underline{g \cdot x \neq x}$

Definitions.

1. isotropy group (stabilizer group

$$\mathrm{Stab}_G(x) := \{ g \in G \,.\, g\cdot x = x \} \subset G$$

$(\equiv G^x)$

$$\left(\begin{array}{l} g_1 \,.\, g_2 \in G^x \\ g_1 \cdot x = x \quad g_2 \cdot (g_1 \cdot x) = g_2 x = x \\ g_2 g_1 \in G^x \end{array} \right)$$

If the group action of G is free

$\iff G^x = \{1\} \quad \forall x \in X.$

2. If $\exists g (\in G^x) \neq 1 \quad g\cdot x = x.$ x is called a fixed point.

$$(X^g \equiv) \, \mathrm{Fix}_X(g) = \{ x \in X : g\cdot x = x \} \subset X$$

is the fixed point set of g.

free $\iff X^g = \phi \quad (g \neq 1)$

3. $O_G(x) = \{ gx. \ \forall g \in G \}$

Theorem (Stabilizer-orbit)

Let X be a G-set. Each left-coset of G^x $(\equiv \mathrm{Stab}_G(x))$ $(x \in X)$ is in a natural

1 - 1 correspondence with points in $O_G(x)$.

There exists a natural isomorphism

$$\varphi : O_G(x) \longrightarrow G/G^x$$

$$g\cdot x \longmapsto g\cdot G^x$$

① Well defined.

$$gx = g'x$$

$$\Leftrightarrow (g'^{-1}g)x = x \Leftrightarrow g'^{-1}g \in G^x \Leftrightarrow gG^x = g'G^x$$

② surjective ✓

injective : $g\cdot G_x = g'\cdot G_x \Rightarrow g x = g' x$

For a finite group : $|O_G(x)| = [G : G^x] = |G| / |G^x|$

Example .

1. G acts on G by conjugation $h \in G$.

$$O_G(h) = \{ghg^{-1}, \forall g \in G\} = C(h)$$

$$Stab_G(h) \equiv G = \{g \in G.\ \underline{ghg^{-1} = h}\} =: C_G(h)$$

Definition . The centralizer of h in G

$$C_G(h) := \{g \in G : gh = hg\}$$

(1) $C_G(h)$ is a subgroup

① $e \in C_G(h)$: $eh = he$

② $\forall g_1, g_2 \in C_G(h)$ $(g_1 g_2^{-1})h = g_1 h g_2^{-1} = h g_1 g_2^{-1}$

$\Rightarrow (g_1 g_2^{-1}) \in C_G(h)$

$|C(h)| = [G : C_G(h)]$

$\uparrow$

number of conjugates of h

extend to subsets:

$C_G(H) = \{ g \in G : gh = hg \quad \forall h \in H \}$

$C_G(G) = Z(G)$