Recap

1. $C(h) := \{ g h g^{-1} : \forall g \in G \}$

2. class function: $f(g g_0 g^{-1}) = f(g_0)$. $\forall g, g_0 \in G$.

   ① character $\chi_T(g) = \mathrm{Tr}\, T(g)$ (T, V)

   ② characteristic polynomial $GL(n, \mathbb{C})$

   $$P_A(x) = \det(x\mathbb{1} - A)$$

   $$P_A = P_{gAg^{-1}}. \quad g \in GL.$$

3. conjugate homomorphisms:

   $$\varphi_i : G_1 \to G_2$$

   $$\varphi_2(g_1) = g_2 \varphi_1(g_1) g_2^{-1} \quad (\forall g_1 \in G_1)$$

   ↪ matrix rep. $\varphi: G \to GL(n, k)$

   conjugate off. $\underline{T_1(g) = S T_2(g) S^{-1}}$ $\forall g \in G$.

4. Conjugacy classes of S_n

   $$(a_1 a_2)(a_3 a_4 a_5) \sim (b_1 b_2)(b_3 b_4 b_5)$$

   $$\tau(a_i) = b_i$$

Young diagram. S^4

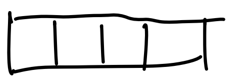
 $= (4)$ Moore's textbooks $(1)^4$

$C(\vec{v}) = (1)^{v_1}(2)^{v_2} \cdots (n)^{v_n}$ $\quad$ $C(\vec{\lambda}) = [\lambda_1, \lambda_2 \cdots \lambda_n]$

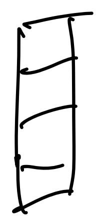
$$\begin{cases} v_1 + v_2 + \cdots + v_n = \lambda_1 \\ v_2 + \cdots + v_n = \lambda_2 \\ \vdots \\ v_n = \lambda_n \end{cases}$$

$$\begin{cases} v_i = \lambda_i - \lambda_{i+1} & (i < n) \\ v_n = \lambda_n \end{cases}$$

$[4]$: $\lambda_1 = 4$, $\lambda_{i \geq 2} = 0$ $\iff$ $v_1 = 4$, $v_{i \geq 2} = 0$ $\quad (1)^4$

 $\lambda_1 = 4$

$\lambda_2 = 0$



$[2^2] = (2)^2$

$[3, 1] = (1)^2(2)$

$\vdots$

I. normal subgroups. $N \triangleleft G$

$$gNg^{-1} = N \quad \forall g \in G.$$

centre $Z(G) = \{ z \in G \mid zg = gz, \ \forall g \in G \}$

6. $N \triangleleft G$. G/N has a natural group structure

$$(g_1 N)\cdot(g_2 N) := g_1 g_2 N.$$

Examples (cont.)

1. $n\mathbb{Z} \triangleleft \mathbb{Z}$

2. $A_3 \triangleleft S_3$

3. $D_4 = \langle a, b \mid a^4 = b^2 = (ab)^2 = 1 \rangle \qquad |D_4| = 8 = 2^3$

$$D_4 = \langle e, a, a^2, a^3, b, ab, a^2b, a^3b \rangle$$

$$\left(\underline{b}a^n = b^{-1}a^n = (ab)^{-1}a^{n+1} = ab\,a^{n+1} = a^2ba^{n+2} \right)$$

non-trivial normal subgroups:

① $\{e, b, a^2b, a^2\} = N_1$

$$a\underline{ba^{-1}} = a\cdot ab = a^2b$$

② $\{e, ab, a^3b, a^2\} = N_2$

$$a(ab)a^{-1} = a^3b$$

③ $\{e, a, a^2, a^3\} = N_3$

④ $\{e, a^2\} = N_4 = Z(G)$

$$\Leftrightarrow a^2b = ba^2$$

other subgroups:

$$\left.\begin{array}{l} \{e, b\} \\ \{e, ab\} \\ \{e, a^2b\} \\ \{e, a^3b\} \end{array}\right\} \simeq \mathbb{Z}_2$$

not normal.

For ①.②.③. $|N_i| = 4$ $|G/N| = 2$ $G/N \cong \mathbb{Z}_2$

① $N_1 = \{e.\ b.\ a^2b.\ a^2\}$

$\cong \begin{cases} \mathbb{Z}_4 \\ V \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \cong D_2 \end{cases}$

$N_1 \cong D_2 \cong V$

$(A = a^2.\ \ B = b \quad \langle A, B \mid A^2 = B^2 = (AB)^2 = 1 \rangle)$

$D_4/N_1 = \{N_1,\ aN_1\} \cong \mathbb{Z}_2 = \{\pm 1\}$

$N_1 \cdot N_1 = N_1 \qquad N_1 \longrightarrow 1$

$N_1 \cdot (aN_1) = aN_1 \qquad aN_1 \longrightarrow -1$

$(aN_1) \cdot (aN_1) = a^2 N_1 = N_1$

	N_1	aN_1
N_1	N_1	aN_1
aN_1	aN_1	N_1

② $N_2 = \{e,\ ab.\ a^2.\ a^3b\} \cong D_2 \quad (A = a^2,\ B = a^3b)$

③ $N_3 = \{e.\ a,\ a^2,\ a^3\} \cong \mathbb{Z}_4$

$D_4/N_3 = \{N_3.\ bN_3\} \cong \mathbb{Z}_2$

④ $N_4 = Z(D_4) = \{e.\ a^2\} \qquad (aZ)\cdot(aZ) = a^2Z = \{a^2, e\}$
$= Z$

$D_4/Z(D_4) = \{Z(D_4),\ aZ(D_4).\ bZ(D_4),\ abZ(D_4)\}$

$\cong D_2$

D_4 is nonabelian. $\Rightarrow$ $D_4/Z(D_4)$ non cyclic

[HW]: $G/Z(G)$ cyclic $\Leftrightarrow$ G is abelian.

4. determinant of A in $GL(n, K)$

$$GL(n, K) \xrightarrow{\det} K$$

$$A \longmapsto \det(A)$$

$[\det(AB) = \det(A)\det(B)]$

$\ker(\det) = SL(n, K)$

$\Rightarrow SL(n, K) \triangleleft GL(n, K)$

$(\det(gAg^{-1}) = \det(A))$

① $GL(n, K)/SL(n, K) \cong K^*$ $\qquad M \in GL$

$\det M = z = re^{i\theta}$

$M = (r^{\frac{1}{n}} e^{i\theta/n}) \cdot A \quad A \in SL$

② $U(n)/SU(n) \cong U(1)$ $\qquad U(n)$: $AA^* = 1$

$|\det A| = 1$

SU: $\det = 1$

③ $O(n)/SO(n) = \{SO(n), PSO(n)\} \cong \mathbb{Z}_2$

$(\det P = -1)$

5. Space group

$$g = \{R_\alpha | \vec{\tau}\} \qquad g \cdot \vec{r} = R_\alpha \cdot \vec{r} + \vec{\tau}$$

$$\{e | \vec{0}\} = \underbrace{\{R_\alpha | \tau\}}_{g} \underbrace{\{R_\beta | \tau'\}}_{g^{-1}} = \{\underbrace{R_\alpha R_\beta}_{e} | \underbrace{R_\alpha \tau' + \tau}_{\vec{0}}\}$$

$$\Rightarrow g^{-1} = \{R_\alpha^{-1} | -R_\alpha^{-1}\tau\}$$

Consider the translation subgroup $T := \langle \vec{t}_1, \vec{t}_2, \vec{t}_3 \rangle$

($\vec{t}_i$ primitive lattice vectors) $\{e | t\} \in T$

$$\{R_\alpha | \tau\}\{e | t\}\{R_\alpha^{-1} | -R_\alpha^{-1}\tau\}$$
$$= \{R_\alpha | \tau\}\{R_\alpha^{-1} | -R_\alpha^{-1}\tau + t\}$$
$$= \{e | R_\alpha(-R_\alpha^{-1}\tau + t) + \tau\}$$
$$= \{e | R_\alpha t\} \in T$$

$$\Rightarrow g T g^{-1} = T \qquad \forall g \in G.$$

$$\Rightarrow T \triangleleft SG$$

6 $\{1\} \triangleleft G$, $G \triangleleft G$ trivial normal subgroups

(Def) A group with no nontrivial normal subgroups is called a <u>simple group</u>.

① $\mathbb{Z}_p \cong \mu_p$ with p prime $\quad H \subset \mathbb{Z}_p \quad |H| = 1$ or p

$H = \{1\}$ or $\mathbb{Z}_p$

② Alternating groups A_n

$A_3 \cong \mathbb{Z}_3$ $\quad$ A_3 is simple

$D_4 \cong V \triangleleft A_4$ $\quad$ A_4 is not simple

$A_{n \geq 5}$ are simple

6.4. Quotient groups and (short) exact sequences

Recall: $K = \ker(\mu)$ the kernel of homomorphism

$$\mu : G \to G'$$

$\Rightarrow K \triangleleft G$

G/K has natural group structure

$$(g_1 K)(g_2 K) := g_1 g_2 K$$

Theorem (1st isomorphism theorem)

$\mu : G \to G'$ homomorphism.

$$\Rightarrow G/K \overset{\mu}{\cong} \operatorname{im}(\mu)$$

Proof. $\varphi : G/K \longrightarrow \operatorname{im}\mu$

$$gK \longmapsto \mu(g)$$

$\varphi(g_1K) = \varphi(g_2K)$

① φ is well-defined. $(g_1K = g_2K \Rightarrow \mu(g_1) = \mu(g_2))$

$g_1K = g_2K \Rightarrow \exists k \in K \;\; g_1 = g_2 k$

$\Rightarrow g_2^{-1} g_1 = k \in K$

$$\Rightarrow \mu(g_2^{-1} g_1) = \mu(g_2)^{-1}\mu(g_1) = 1_{G'}$$

$$\Rightarrow \mu(g_1) = \mu(g_2)$$

② φ is a homomorphism.

$$\underline{\varphi(g_1 K \cdot g_2 K)} = \varphi(g_1 g_2 K) = \mu(g_1 g_2)$$

$$= \mu(g_1)\mu(g_2) = \underline{\varphi(g_1 K)\varphi(g_2 K)}$$

a. $\operatorname{im}\varphi = \operatorname{im}\mu$ surjective

b. $\varphi(g_1 K) = \varphi(g_2 K) \overset{!}{\Leftrightarrow} \mu(g_1) = \mu(g_2)$ injective

RHS $\Leftrightarrow \mu(g_1 g_2^{-1}) = 1_{G'}$

$\Rightarrow g_1 g_2^{-1} \in K$

$\Rightarrow g_1 K = g_2 K$

a+b: φ is an isomorphism.

Example. homomorphism

$$\pi : SU(2) \longrightarrow SO(3)$$

$$u\,\vec{x}\cdot\vec{\sigma}\,u^{+} := (\pi(u)\vec{x})\cdot\vec{\sigma}$$

$u \in \ker\pi$. $u\,\vec{x}\cdot\vec{\sigma}\,u^{+} = \vec{x}\cdot\vec{\sigma}$ $\quad u = \lambda \mathbb{1}$

$\lambda = \pm 1$

$$\Rightarrow \ker\pi \cong \mathbb{Z}_2$$

$$SU(2)/\mathbb{Z}_2 \cong SO(3)$$

Now we introduce a sequence of homomorphisms

$$\cdots \; G_{i-1} \xrightarrow{f_{i-1}} G_i \xrightarrow{f_i} G_{i+1} \xrightarrow{f_{i+1}} \cdots$$

The sequence is exact at G_i if

$$\operatorname{im} f_{i-1} = \ker f_i$$

A short exact sequence 正合序列 (SES) is of the form

$$1 \longrightarrow G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \xrightarrow{f_3} 1$$

$0 \qquad\qquad\qquad\qquad 0$

① 1 represents trivial group. $\{1\}$

0 : abelian groups "$+$" as group multiplication

② $1 \to G_1$: inclusion map. } unique

$G_3 \to 1$: trivial homomorphism

Exactness at G_i :

1. G_1 : $\ker f_1 = \{1_{G_1}\} \Rightarrow f_1$ is injective
2. G_2 : $\ker f_2 = \operatorname{im} f_1$
3. G_3 : $\ker f_3 = G_3 = \operatorname{im} f_2 \Rightarrow f_2$ is surjective

Now consider a homomorphism $\mu: G \to G'$

$K = \ker \mu$.

We have

inclusion map

$$1 \longrightarrow K \xhookrightarrow{i} G \xrightarrow{\mu} \operatorname{im} \mu \longrightarrow 1$$

$$\operatorname{im}\mu \cong G/K$$

Exactness check:

1. K. $\ker i = \{1_G\}$ ✓

2. G: $\ker \mu = \operatorname{im} i = K$ ✓

3. $\operatorname{im}\mu$: $\ker(\operatorname{im}\mu \to 1) = \operatorname{im} \mu$ ✓

1st isomorphism theorem $\Rightarrow$

$$\boxed{1 \longrightarrow K \longrightarrow G \longrightarrow G/K \longrightarrow 1}$$

<u>Remarks</u>.

1. If we have SES.

$$1 \to N \to G \to Q \to 1$$

then $N \cong H \trianglelefteq G$ (it's the kernel of homomorphism $G \to Q$)

We sometimes write Q as $G/f(N)$

where $f: N \xrightarrow{f} G$ is an injective homomorphism.

"G is an extension of Q by N"

Example.

1. $$1 \to G_1 \to G_1 \times G_2 \to G_2 \to 1$$
$$\quad (G_2) \qquad\qquad\qquad (G_1)$$

$$\mu: G_1 \times G_2 \to G_2 \qquad \begin{pmatrix} g_1 \in G_1 \\ g_2 \in G_2 \end{pmatrix}$$
$$(g_1, g_2) \mapsto g_2$$

2. $\varphi: \mu_4 \to \mu_2 \qquad (\mathbb{Z}_4 \to \mathbb{Z}_2)$
$\quad w \mapsto w^2 \qquad w = e^{i\frac{2\pi}{4}}$

$$\ker \varphi = \{\pm 1\} \cong \mathbb{Z}_2$$

$$1 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 1$$

in general $1 \to \mathbb{Z}_n \to \mathbb{Z}_{n^2} \to \mathbb{Z}_n \to 1$

$$\left(\begin{array}{l} \varphi: \mu_{n^2} \to \mu_n \\ \quad z \mapsto z^n \end{array} \right)$$