Recap.

1. sgn: $S_n \longrightarrow \mathbb{Z}_2$

$$\phi \longmapsto sgn(\phi) \in \{\pm\}$$

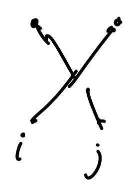
$$sgn(\phi_1 \cdot \phi_2) = sgn(\phi_1)\, sgn(\phi_2)$$

& $sgn(1) = 1$

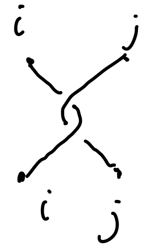
$$\ker(sgn) = A_n$$

2. σ_i vs $\tilde{\sigma}_i$

S_n $\qquad$ B_n



$\tilde{\sigma}^2$



$\neq 1$

3. view from group actions.

set X. $\quad$ G embed S_X

Cosets: $\quad$ H acts on G. $\quad$ (H subgroup of G)

orbit, $gH := \{gh \mid h \in H\}$

$|gH| = |H| \implies |H| \,\big|\, |G|$ $\quad$ (Lagrange)

$X/\sim \implies G/H$: set of cosets

$|G/H| := [G : H] = |G|/|H|$ $\quad$ index of H

4. Conjugacy. (G acts on G by conjugation)

$$C(h) := \{ g h g^{-1} ; \forall g \in G \}$$

<u>Example</u>.

1. Permutations ϕ_1, ϕ_2 are conjugate if they have the same cycle decomposition structure.

$$(a_1 a_2)(a_3 a_4 a_5) \sim (b_1 b_2)(b_3 b_4 b_5)$$

$\tau = (i\ a_1 a_2 \cdots a_m\ j\ b_1 \cdots b_n)$

$\tau(i) = a_1 \quad \tau(j) = b_1$

$\tau(ij)\tau^{-1} = (i\ a_1 a_2 \cdots a_m\ j\ b_1 \cdots b_n)(ij)(b_n \cdots b_1\ j\ a_m \cdots a_1\ i)$

$\begin{cases} i \mapsto b_n \mapsto b_n \mapsto i \\ j \mapsto a_m \mapsto a_m \mapsto j \end{cases}$ $\qquad \begin{cases} a_i \mapsto a_{i-1} \mapsto a_{i-1} \mapsto a_i \\ b_i \longrightarrow b_i \end{cases}$

$\begin{cases} a_1 \mapsto i \mapsto j \mapsto b_1 \\ b_1 \mapsto j \mapsto i \mapsto a_1 \end{cases}$

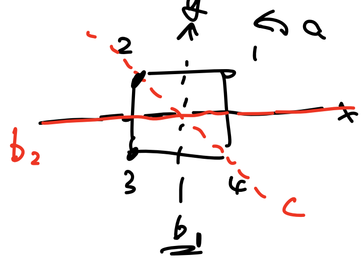
$\tau(ij)\tau^{-1} = (a_1 b_1) = (\tau(i), \tau(j))$

$\Rightarrow \tau(a_1 a_2 \cdots a_n)\tau^{-1} = (\tau(a_1), \tau(a_2), \cdots \tau(a_n))$

$\tau(a_1 a_2)(a_3 a_4 a_5)\tau^{-1} = (b_1 b_2)(b_3 b_4 b_5)$

$\Leftrightarrow \boxed{\tau(a_i) = b_i}$

2. $D_4 := \langle a, b : a^4 = b^2 = 1, (ab)^2 = 1 \rangle$



$a = (1234)$

$b_1 = (12)(34)$

$c = ab = (1234)(12)(34) = (13)(2)(4) = (13)$

$c b_1 c^{-1} = (13)(12)(34)(13) = (14)(23) = b_2$ $\quad b_1 \sim b_2$

$D_4 = \{1\} \cup \{a^2\} \cup \{a, a^3\} \cup \{b, a^2 b\} \cup \{ab, a^3 b\}$

$= \{()\} \cup \{(13)(24)\} \cup \{(1234), (1432)\}$

$\cup \{(12)(34), (14)(23)\} \cup \{(13), (24)\}$

$\Big(\tau (13)(24) \tau^{-1} = (12)(34)$

$\tau(3) = 2 \quad \tau(2) = 3 \qquad \tau = (23) \Big)$

3. in $GL(n, K)$:

$G = U(n) := \{ A \in M_n(\mathbb{C}) \mid AA^{\dagger} = 1_n \}$

Spectral theorem ensures $u \in U(n)$ can be diagonalized as. $\exists g \in U(n)$

$g u g^{-1} = \mathrm{diag}(z_1, \cdots z_n) \quad (|z_i| = 1)$

$\hookrightarrow$ conjugacy classes labeled by $(z_1 \cdots z_n)$? $\nearrow U(1)^n$

permutation $A(\phi)\, \mathrm{diag}(z_1, \cdots z_n) A(\phi)^{-1}$

$= \mathrm{diag}(z_{\phi(1)}, z_{\phi(2)}, \cdots z_{\phi(n)})$

$$[A(\phi)g]^{u}[A(\phi)g]^{-1} = \mathrm{diag}\{z_{\phi(i)}\}$$

$\Rightarrow$ $U(1)^n/S_n$ labels conj. class.

4. a general element of $GL(n,\mathbb{C})$ is not diagonalizable. Define the characteristic polynomial ($A \in GL(n)$)

$$P_A(x) = \det(x\mathbb{1} - A)$$

$$\begin{aligned} P_{gAg^{-1}}(x) &= \det(x\mathbb{1} - gAg^{-1}) \\ &= \det(g(x\mathbb{1} - A)g^{-1}) \\ &= \det(x\mathbb{1} - A) = P_A(x) \end{aligned}$$

<u>Definition</u> A <u>class function</u> on a group is a function f on G, s.t.

$$f(gg_0g^{-1}) = f(g_0) \qquad \forall g, g_0 \in G.$$

For a matrix representation, define the <u>character</u> of the representation

$$\chi_T(g) := \mathrm{Tr}\, T(g)$$

It is a class function.

<u>Definition</u> Two homomorphisms $\varphi_i : G_1 \to G_2$ are conjugate if $\exists g_2 \in G_2$, s.t.

$$\varphi_2(g_1) = g_2 \varphi_1(g_1) g_2^{-1}$$

in terms of representations $(T: G \to GL(V))$

$$\begin{array}{ccc} V_1 & \xrightarrow{S} & V_2 \\ {\scriptstyle T_1(g)}\downarrow & & \downarrow{\scriptstyle T_2(g)} \\ V_1 & \xrightarrow{S} & V_2 \end{array}$$

(equivariant map / morphism of G-space)

$T_2(g) S = S T_1(g)$ $(\dim V_1 = \dim V_2)$

$T_2(g) = S T_1(g) S^{-1}$ ⇐ equivalent representation

5. conjugacy classes in S_n.

Permutations with same structure of cycle decomposition are conjugate.

The conjugacy classes are labeled by the cycle decomposition of their elements. $C(\vec{l})$

$\vec{l} = (l_1, l_2, \cdots, l_n)$ where l_r is the number of r-cycles.

$$n = \sum_{j=1}^{n} j \cdot l_j$$

$\phi = (12)(34)(678)(11,12) \in S_{12}$

$= (12)(34)(5)(678)(9)(10)(11,12)$

$\vec{l} = \begin{matrix} l_1 & l_2 & l_3 & l_{\geq 4} \\ 3, & 3 & 1 & 0 \end{matrix}$ $\qquad \vec{l} = (3, 3, 1, 0, \cdots 0)$

$\Rightarrow$ The number of conjugacy classes of S_n is given by the partition function of n.

__$p(n)$__. namely the number of distinct partitions of n into sum of nonnegative integers.

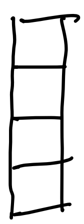
__Example__ S_4

partition	cycle decomp.	typical g	$\lvert C(g)\rvert$	order of g
$4 = 1+1+1+1$	$(1)^4$	1	1	1
$4 = 1+1+2$	$(1)^2(2)$	(ab)	$\binom{4}{2} = 6$	2
$4 = 1+3$	$(1)(3)$	(abc)	$2\binom{4}{3} = 8$	3
$4 = 2+2$	$(2)^2$	$(ab)(cd)$	$\frac{1}{2}\binom{4}{2} = 3$	2
$4 = 4$	(4)	$(abcd)$	6	4

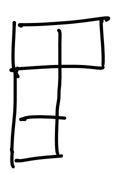
$|S_4| = 24 = 1 + 6 + 8 + 3 + 6$ $\qquad p(4) = 5$

$\boxed{\lambda_i = \sum_{i=i}^{n} l_j}$

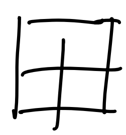
Young diagram :



$[1^4]$

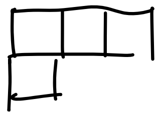
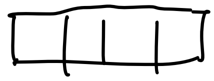


$[2, 1^2]$



$[2^2]$

λ_i : number of boxes in i-th row

 [3,1]  [4] $\lambda_i \geq \lambda_{i+1}$.

Example in physics: a collection of harmonic oscillators $h_j = \hbar \omega_j (a_j^{\dagger} a_j + \frac{1}{2})$ $(\omega_j = j\,\omega_0)$

$$H = \sum_{j=1}^{n} \hbar \omega_j (a_j^{\dagger} a_j + \frac{1}{2})$$

fixed total E, $E = \underline{N}\,\hbar\omega_0$

$$|\phi_{\vec{l}}\rangle = \frac{1}{\sqrt{l_1!\, l_2! \cdots l_n!}} (a_1^{\dagger})^{l_1} (a_2^{\dagger})^{l_2} \cdots (a_n^{\dagger})^{l_n} |0\rangle$$

$$n = \sum_{j=1}^{n} j\, l_j$$

p(n) is the degeneracy of states

— 6.3. Normal subgroups & Quotient groups

Defition A subgroup $N \subset G$ is called a normal subgroup or an invariant subgroup if

$$gNg^{-1} = N \quad \forall g \in G.$$

denoted $N \lhd G$.

* NB. it doesn't mean $gng^{-1} = n \ \forall n \in N$!

Suppose a subgroup Z satisfies

$$gzg^{-1} = z \quad \forall z \in Z \quad \forall g \in G.$$

$$Z(G) := \{ z \in G \mid zg = gz. \ \forall g \in G \}$$

$Z(G)$ is an abelian normal subgroup of G.

$Z(G)$ is the center of G.

Examples.

1. G is abelian. all subgroups are normal.

$$ghg^{-1} = (gg^{-1})h = h \quad \forall h \in G.$$

2. The kernel of a homomorphism

$$\phi : G \longrightarrow G'$$

is a normal subgroup.

$k \in \ker(\phi)$. $\phi(k) = 1_{G'}$

$$\phi(gkg^{-1}) = \phi(g)\cancel{\phi(k)}\phi(g^{-1}) = \phi(g)\phi(g)^{-1} = 1 \quad (\forall g \in G)$$

$\Rightarrow gkg^{-1} \in \ker(\phi)$

$\Rightarrow \ker\phi \lhd G$

<u>Theorem</u>. If $N \lhd G$. then the set of left cosets $G/N = \{gN, g \in G\}$. has a <u>natural</u> group structure with group multiplication defined as

$$(g_1N)\cdot(g_2N) := (g_1g_2)N$$

We call the groups of the form G/N <u>quotient groups</u>. (factor groups)

$$g_1N \cdot g_2N = g_1(g_2g_2^{-1})Ng_2N$$
$$= g_1g_2\underbrace{(g_2^{-1}Ng_2)}_{=N}N$$
$$= g_1g_2N$$

<u>Corollary</u>. If $N \lhd G$. then the <u>natural map</u>

$$\phi : G \longrightarrow G/N$$
$$g \longmapsto gN$$

is a surjective homomorphism. <u>$\ker\phi = N$</u>

$\phi(g_1)\phi(g_2) = g_1N \cdot g_2N = g_1g_2N = \phi(g_1g_2)$

$g \in \ker\phi \quad \phi(g) = gN = N \iff g \in N$

Every normal subgroup is the kernel of some homomorphism.

Example.

1 $n\mathbb{Z} := \langle n \rangle \triangleleft \mathbb{Z}$

$= \{ \cdots -2n, -n, 0, n, 2n, \cdots \}$

$\mathbb{Z}/n\mathbb{Z} := \{ i + n\mathbb{Z},\ 0 \le i \le n-1 \}$

$\phi : \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z}$

$i \longmapsto i + n\mathbb{Z}$

$\ker\phi = n\mathbb{Z}$

$\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$

quotient groups are not subgroups

(special cases. e.g

$\mathbb{Z}_2 \triangleleft \mathbb{Z}_4$

$\mathbb{Z}_4/\mathbb{Z}_2 \cong \mathbb{Z}_2$)

2. $A_3 \triangleleft S_3 \qquad \phi : S_3 \longrightarrow \mathbb{Z}_2 \qquad \ker(\phi) = A_3$

[HW] $H \subset G$. $[G : H] = 2 \Rightarrow H \triangleleft G$