Recap:

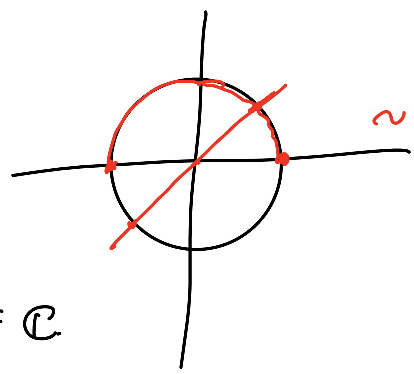
1. $GL(V) \overset{u}{\cong} GL(n,k)$ basis for group presentation

2. G-set: set with group action.

3. orbits. $x \in G$-set X

$$O_G(x) := \{ g\cdot x \mid \forall g \in G \}$$

4. $\mathbb{R}P^{n-1} = \mathbb{R}^n - \{\vec{0}\} / \mathbb{R}^*$



$\mathbb{R}P^1 \cong S^1$

$$\boxed{K^n - \{\vec{0}\} / K^*}$$

$K = \mathbb{C}$

$\mathcal{H} = \mathbb{C}^2$ $\quad |\varphi\rangle = z_1\begin{pmatrix}1\\0\end{pmatrix} + z_2\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}z_1\\z_2\end{pmatrix} \in \mathcal{H}.$

$H|\varphi\rangle = E|\varphi\rangle \quad H(\lambda|\varphi\rangle) = E(\lambda|\varphi\rangle) \quad \lambda \in \mathbb{C}^*$

$(\mathbb{C}^2 - \vec{0})/\mathbb{C}^* = \mathbb{C}P^1$

$\lambda\begin{pmatrix}z_1\\z_2\end{pmatrix} \quad \underline{z_1/z_2} \quad \left(z_1'/z_2' = \lambda z_1/\lambda z_2 = z_1/z_2 \right)$

$\begin{pmatrix}1\\0\end{pmatrix} \to \infty$

"complex plane + infinity": $\hat{\mathbb{C}}$

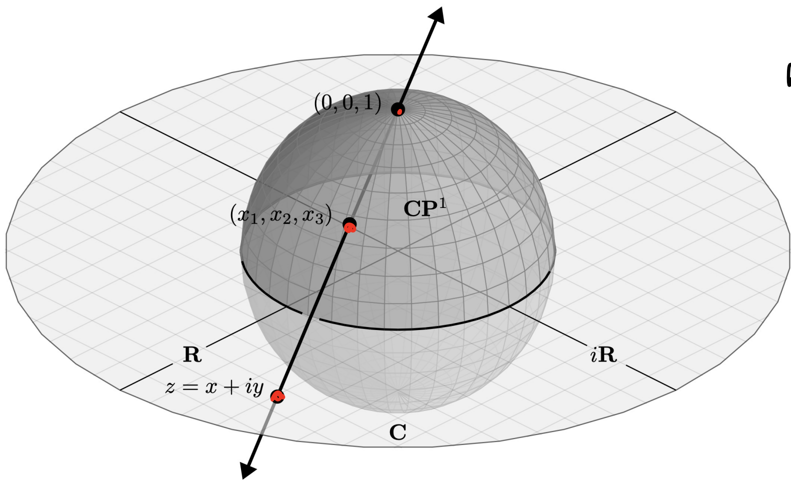


Figure 7.2: The complex projective line $\mathbf{CP}^1$.

Riemann sphere

$$\begin{cases} x = \dfrac{x_1}{1-x_3} \\ y = \dfrac{x_2}{1-x_3} \end{cases}$$

$$CP^1 \cong S^2$$

$$\begin{cases} x_1 = \dfrac{2x}{x^2+y^2+1} \\ x_2 = \dfrac{2y}{x^2+y^2+1} \\ x_3 = \dfrac{x^2+y^2-1}{x^2+y^2+1} \end{cases}$$

Definition Let X, X' be two G-spaces

A <u>equivariant map</u>, $f: X \to X'$

satisfies

$$f(g \cdot x) = g \cdot f(x) \qquad \forall x \in X \ \forall g \in G.$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\Phi(g)"'] & X' \arrow[d, "\Phi'(g)"] \\ X \arrow[r, "f"] & X' \end{tikzcd}$$

$$f(\phi(g.x)) = \Phi'(g. f(x))$$

f is also called a morphism of G-spaces.

Examples.

1. $G = \mathbb{Z}_2$ on R^{n+1}

$$\sigma = \left(\begin{array}{c|c} 1_p & 0 \\ \hline 0 & -1_q \end{array}\right) \qquad p+q = n+1$$

$\sigma(\vec{x}) = (x_1 \cdots x_p, -x_{p+1} \cdots -x_{n+1})$

$$\begin{tikzcd} R^{n+1} \arrow[r, "M"] \arrow[d, "\sigma"'] & R^{n+1} \arrow[d, "\sigma"] \\ R^{n+1} \arrow[r, "M"] & R^{n+1} \end{tikzcd}$$

$\sigma(\underline{M} \cdot \vec{x}) = M(\sigma \cdot \vec{x})$

$\Rightarrow M\sigma = \sigma M$

$$M = \begin{array}{cc} p & q \\ \end{array}$$
$$M = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right)$$

2. $G = \mathbb{Z}$ on $\mathbb{R}$

$$\phi_n : \quad x \to x+n \quad (n \in \mathbb{Z})$$

$$f : \mathbb{R} \to \mathbb{R}$$

$$\begin{tikzcd} \mathbb{R} \arrow[r, "f"] \arrow[d, "\phi_n"'] & \mathbb{R} \arrow[d, "\phi_n"] \\ \mathbb{R} \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

$$f(x) + n_1 = f(x+n_1)$$

$$f(x) + n_2 = f(x+n_2)$$

$$f(\underline{x}+n_1) - f(\underline{x}+n_2) = \underline{n_1 - n_2}$$

$$f(x) = x + \alpha$$

5. The symmetric group

Recall that

Given a set X, the set of permutations

$$S_X := \{ X \xrightarrow{f} X \ : \ f : 1\text{-}1 \ \& \text{ onto (invertible)} \}$$

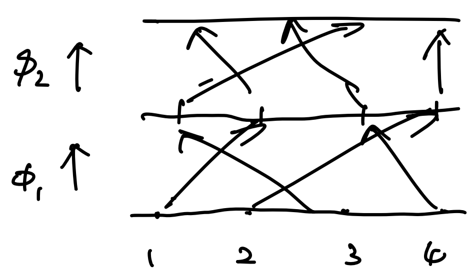
For $n \in \mathbb{N}^+$ denote the symmetric group on n elements, S_n, which is the set of all permutations of the set $X = \{1, 2, \cdots, n\}$

$(|S_n| = n!)$

A permutation can be written as

$$\phi = \begin{pmatrix} 1 & 2 & \cdots & n \\ p_1 & p_2 & \cdots & p_n \end{pmatrix} \quad \text{with } p_i = \phi(i)$$

$$\phi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$$



$$\phi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$$

$$\phi_2 \cdot \phi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$$

$$\phi_2 \cdot \phi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 4 & 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} = (2\ 4)$$

<u>Definition</u>. Let $i_1, \cdots i_r$ be distinct integers between 1 and n.

If $\phi \in S_n$ fixes the remaining integers and if

$$\phi(i_1) = i_2, \quad \phi(i_2) = i_3, \quad -- \quad \phi(i_r) = i_1$$

then ϕ is an <u>r-cycle</u> (cycle of length r)

$$(i_1\, i_2\, i_3 \cdots i_r)$$

A 2-cycle is called a transposition.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \\ 2 & 3 & 4 & 1 \end{pmatrix} = (1234)$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 4 & 2 & 3 \end{pmatrix} = (1)(243) = (243)$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & & & \\ 2 & 1 & 4 & 3 \end{pmatrix} = (12)(34)$$

Remarks:

1. cycles are the same up to cyclic ordering

   $(234) = (423) = (342)$

2. disjoint cycles commute

   $(234)(56) = (56)(236)$

   $(12)(23) \neq (23)(12)$

3. inverse of a permutation

   $[(12)(345)]^{-1} = (12)(543) = (12)(354)$

Theorem: Every permutation $\phi \in S_n$ is either a cycle or can be factorized into disjoint cycles.

(Proof by induction)

(Def) complete factorization: is a product of disjoint cycles which contains one 1-cycle for each fixed x.

$(1)(234)$ $\left(\neq (1)(1)(234)\right)$

Complete factorization of a permutation ϕ is unique (up to ordering), which we call the <u>cycle decomposition</u> of ϕ.

Theorem (Cayley, 1878)

Every group G is isomorphic to a subgroup of S_G ("can be embeded in S_G)

In particular, if $|G| = n$, then G is isomorphic to a subgroup of S_n.

($S_G \cong S_N$ with an <u>ordered set</u>.

$\{1, \omega, \omega^2, \cdots \omega^{n-1}\} =: \mu_N$ $\quad S_{\mu_N} \cong S_N$

"natural ordering"

D_n, $SU(n)$. has no natural order)

Example. $Z_n \cong \langle (12 \cdots n) \rangle \cong \mu_N$

$n = 3$

$\langle (123) \rangle = \{1, (123), (132)\} = A_3 < S_3$

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

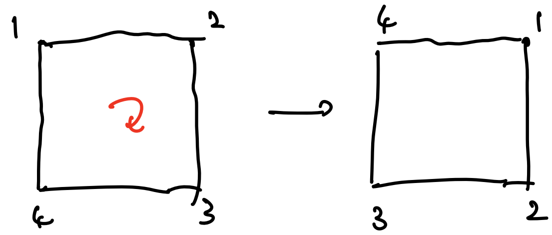
$\mu_3 \quad \{1, \quad \omega \quad \omega^2\}$

Example. $D_4 = \langle A B \mid A^4 = B^2 = (AB)^2 = 1 \rangle$

$|D_4| = 8 \cong$ a subgroup of S_8

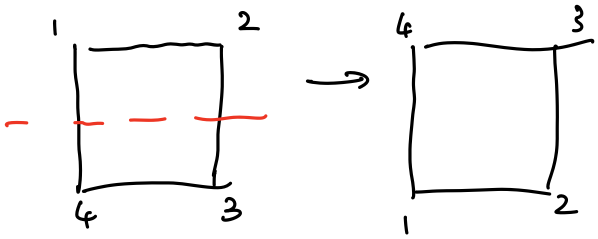
subgroup of S_4

A :



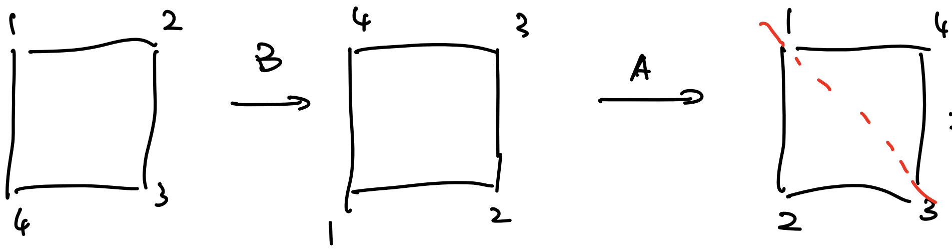
$:= (1\,2\,3\,4)$ 4 $A^4 = 1$

B:



$:= (14)(23)$ 2 $B^2 = 1$

AB:



$:= (24)$

$(AB)^2 = 1$

How to find the isomorphism?

→ use multiplication table (Cayley table)

Klein's 4-group. $V = \langle a b \mid a^2 = b^2 = (ab)^2 = e \rangle$
$\cong \mathbb{Z}_2 \times \mathbb{Z}_2$

$e = (0, 0)$

$a = (1, 0)$

$b = (0, 1)$

$c = (1, 1)$

$|V| = 4$

$$\phi : V \longrightarrow \operatorname{im}(V) \subset S_4$$
$$a \longmapsto \phi(a)$$

		e	a	b	c
1	e	e	a	b	c
2	a	a 2	e 1	c 4	b 3
3	b	b 3	c 4	e 1	a 2
4	c	c	b	a	e

$$\phi(e) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = 1$$

$$\phi(a) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = (12)(34)$$

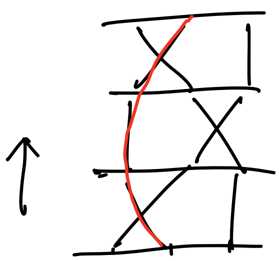
$$\phi(b) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = (13)(24)$$

$$\phi(c) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = (14)(23)$$

2-cycles / transpositions.

i, j, k are distinct.

① $(ij)(jk)(ij) = (ik) = (jk)(ij)(jk)$



$i \quad j \quad k$

② $(ij)^2 = 1 \qquad (ij) = (ij)^{-1}$

③ $(ij)(kl) = (kl)\cdot(ij) \qquad \{i, j\} \cap \{k, l\} = \phi$

Theorem. Every permutation $\phi \in S_n$ is a product of transpositions.

Proof. $\phi \in S_n$ has a cycle decomposition.

For each cycle,

$$(1\,2 \cdots r) = (1\,r)(1\,r-1) \cdots (1\,2)$$

transpositions generate the permutation group.

Remarks:

1. There are other ways to generate S_n

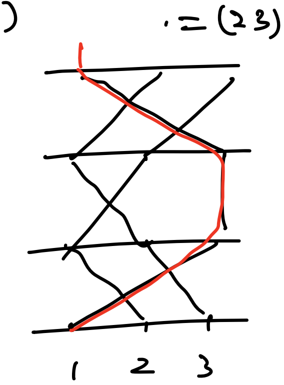
① $\sigma_i = (i\ \ i+1)$ $\quad (1 \le i \le n-1)$

"elementary generators"

$$(i\,j) = (i,\, i+1)(i+1,\, j)(i,\, i+1) \qquad (i < j)$$

② generated by $(1\,2)$ & $(1\,2 \cdots n)$

$$(2\,3) = (1\,2 \cdots n)(1\,2)(1 \cdots n)^{-1}$$



Remark: transposition decomposition is not unique

$$(123) = \underbrace{(13)(12)}_{2} = \underbrace{(23)(13)}_{2}$$

$$= \underbrace{(13)(42)(12)(14)}_{4}$$

$$= \underbrace{(13)(42)(12)(14)(23)(23)}_{6} \cdots$$

<u>Definition</u> A permutation $\phi \in S_n$ is even (odd) if it is a product of even (odd) transpositions. ("Parity")

<u>Definition</u>. If $\phi = \sigma_1 \cdots \sigma_t$ ($\in S_n$) is a complete cycle decomposition.

$$sgn(\phi) = (-1)^{n-t}$$

cycle decomp. is unique $\Rightarrow$ sgn is well-defined.

$(123) \in S_3$

$$sgn((123)) = (-1)^{3-1} = 1 \quad \text{even.}$$