Recap

1. $G = \langle x \rangle$

$\mathbb{Z}_n = \langle A \mid A^n = 1 \rangle \cong \mu_n$

$V \cong D_2 \quad \langle A, B \mid A^2 = B^2 = (AB)^2 = 1 \rangle$

2. homomorphism / isomorphism.

$$\varphi : G \to G'$$

$$\varphi(g_1 \cdot g_2) = \varphi(g_1) \cdot \varphi(g_2)$$

$$\begin{tikzcd} G \times G \arrow[r, "m"] \arrow[d, "\varphi \times \varphi"'] & G \arrow[d, "\varphi"] \\ G' \times G' \arrow[r, "m'"] & G \end{tikzcd}$$

$\varphi : G \to H$

3. $\ker \varphi = \{ g \in G : \varphi(g) = 1_H \} \subset G$

$\operatorname{im} \varphi = \varphi(G) \subset H$

4. $SU(2) \longleftrightarrow SO(3)$

$R : SU(2) \to GL(3, \mathbb{R})$

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "R(u)"] \arrow[d, "h"'] & \mathbb{R}^3 \arrow[d, "h"] \\ \mathcal{H}_2^0 \arrow[r] & \mathcal{H}_2^0 \end{tikzcd}$$

$C_u \quad (u \in SU(2))$

$h \cdot R(u) = C_u \cdot h$

$(R(u) \cdot \vec{x}) \cdot \vec{\sigma} = u \, \vec{x} \cdot \vec{\sigma} \, u^{-1}$

$\ker R = \{\pm 1\} \cong \mathbb{Z}_2$ $\quad R(u) \in SO(3)$

$R(u) = R(-u)$

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①

Example. $GL(V)$ and $GL(n, K)$

Let $GL(V): V \to V$ be the group of invertible linear transformations with a finite dimensional vector space V.

Given an ordered basis $b = \{\hat{e}_1, \dots \hat{e}_n\}$

Define a homomorphism:

$$\varphi_b : GL(V) \longrightarrow GL(n, K)$$

$$\tau \longmapsto T_b(\tau)$$

s.t. $\underline{\tau(\hat{e}_i) = \sum_j \hat{e}_j \cdot T_b(\tau)_{ji}}$ ⓐ

$\forall \vec{v} \in V$. $\vec{v} = \sum_{i=1}^n v_i \hat{e}_i$ $\quad (v_i \in K)$

$$\tau \vec{v} = \sum_{i=1}^n v_i (\tau \hat{e}_i) = \sum_{ij} \hat{e}_j \cdot T_b(\tau)_{ji} v_i$$

$$\Rightarrow \tau_1(\tau_2 \vec{v}) = \sum_{ij} (\tau_1 \cdot \hat{e}_j) T_b(\tau_2)_{ji} v_i$$

$$= \sum_{ijk} \hat{e}_k \cdot T_b(\tau_1)_{kj} T_b(\tau_2)_{ji} v_i$$

$$= \sum_{ik} \hat{e}_k [T_b(\tau_1) T_b(\tau_2)]_{ki} v_i$$

$$= (\tau_1 \tau_2) \vec{v}$$

$$= \sum_{ik} \hat{e}_k [T_b(\tau_1\tau_2)]_{ki} v_i$$

$$\Rightarrow T_b(\tau_1\tau_2) = T_b(\tau_1)T_b(\tau_2)$$

surjective ✓

injective ? $\tau(\hat{e}_i) = e_i \Leftrightarrow \tau = id$ ✓

$$\Updownarrow$$

$$T_b(\tau) = \mathbb{1}_n$$

isomorphism $GL(V) \cong GL(n, K)$

Definition.

① Let G be a group. then a finite dimensional representation of G is a finite dimensional vector space V with a group homomorphism

$\varphi : G \longrightarrow GL(V)$

V: carrier space

② A matrix representation of G is a homomorphism

$$\varphi : G \longrightarrow GL(n, K) \qquad (K = \mathbb{R}, \mathbb{C})$$

$$g \longmapsto P(g)$$

$$\forall g_1, g_2 \in G : \quad P(g_1 g_2) = P(g_1)P(g_2)$$

① + an ordered basis $\longrightarrow$ ② $(GL(V) \cong GL(n, K))$

Matrix rep. is basis dependent

$$\hat{e}_i = \sum_{j=1}^{n} S_{ji}\,\hat{e}'_j$$

$$\Gamma'(g) = S\,\Gamma(g)\,S^{-1}$$

Definition (equivalent representation) Γ, Γ' are

n-dim reps of G

Γ, Γ' are equivalent ($\Gamma \cong \Gamma'$) if $\exists\, S \in GL(n,k)$

s.t. $\forall g \in G \quad \Gamma'(g) = S\,\Gamma(g)\,S^{-1}$

Example $\mathbb{Z}, \mathbb{R}, \mathbb{C} \ni a$

$$\Gamma(a) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$$

$$\Gamma(a)\Gamma(b) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a+b & 1 \end{pmatrix}$$

Example. $S_2 = \{e, \sigma\} \qquad \sigma^2 = e$

$$S_2 \cong \mu_2 \cong \mathbb{Z}_2$$

$$\Gamma(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Gamma(\sigma) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Gamma(\sigma^2) = \Gamma(\sigma)\cdot\Gamma(\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

<u>Example</u> $\mu_3 = \langle \omega \mid \omega^3 = 1 \rangle$

$$\Gamma(e) = \mathbb{1}_3$$

$$\Gamma(\omega) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\Gamma(\omega^2) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

<u>Example</u> $D_4 = \langle a, b \mid a^4 = b^2 = (ab)^2 = 1 \rangle$ ✓

$$|D_6| = 8$$

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$C = AB = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Gamma(D_6) = \{ \pm \mathbb{1}, \pm A, \pm B, \pm C \}$$

isomorphism : "faithful representation"

not faithful. $\Gamma(A) = \Gamma(B) = \mathbb{1}$

4. Group actions on sets

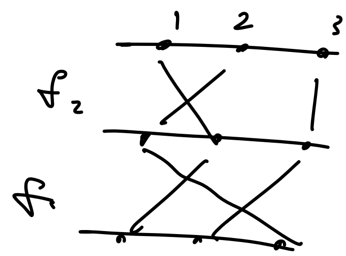
<u>Definition</u>: Given a set X, the set of permutations

$$S_X := \{ X \xrightarrow{f} X \; : \; f : 1-1 \ \& \text{ onto (invertible)} \}$$

is a group under composition.

$$m(f_1, f_2) := f_1 \circ f_2$$

$$\begin{tikzcd} X \arrow[r, shift left, "f_2"] \arrow[rr, bend right, "f_1 \circ f_2"'] & X \arrow[l, shift left, "f_2"] \arrow[r, shift left, "f_1"] & X \arrow[l, shift left, "f_1^{-1}"] \end{tikzcd}$$



<u>Definition</u>. A (left) group action Φ of G is a homomorphism

$$\Phi : G \longrightarrow S_X$$
$$g \longmapsto \underline{\underline{\phi(g, \cdot)}}$$

$\phi(g, \cdot) : X \to X$

$x \mapsto \phi(g, x)$

1. $\phi : G \times X \longrightarrow X \qquad \phi(g, x) \in X \quad (\forall x \in X)$

$$\phi(g_1, \phi(g_2, x)) = \phi(\underline{g_1 g_2}, x)$$

$$\phi(1_G, x) = x \quad (\forall x \in X)$$

$$\phi(g, \phi(g^{-1}, x)) = \phi(g g^{-1}, x) = \phi(1_G, x) = x$$

simplified notation: $g \cdot x := \phi(g, x)$

$$g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x \quad (\forall x \in X)$$

<u>Definition</u> : If a set X has a group action by G we say that X is a <u>G-set</u>.

<u>Example</u> 1. $X = G$.

① group action by <u>multiplication</u>

$x \in X = G$

$g_1 \cdot (g_2 x) = g_1 g_2 x = (g_1 g_2) x$

② group action by <u>conjugation</u>

$g \cdot x := g x g^{-1} \in G = X$

a. $g_1 \cdot (g_2 \cdot x) = g_1 (g_2 x g_2^{-1}) = g_1 g_2 x g_2^{-1} g_1^{-1}$

$= (g_1 g_2) \cdot x$

b. $e \cdot x = e x e^{-1} = x$

Abelian group. $g \cdot x = g x g^{-1} = x \quad (\forall g \in G)$

2. $GL(n, k)$ acts on k^n.

$A \cdot \vec{v} = \sum_j A_{ij} v_j$

$e = 1_n$

a rep. of G. $\Rightarrow$ group action on carrier space V.

3. Space group action on $\mathbb{R}^3$

$\{g \mid \tau\}$ $\qquad g \in O(3)$

$\tau \in T$ translation

$$\{R_g \mid \vec{\tau}\} \cdot \underline{\vec{r}} := R_g \vec{r} + \vec{\tau} \qquad R_g \in O(3)$$

$$\underline{\{R_1 \mid \vec{\tau}_1\}\{R_2 \mid \vec{\tau}_2\} \cdot \vec{r}} = \{R_1 \mid \vec{\tau}_1\}(R_2 \vec{r} + \vec{\tau}_2)$$

$$= R_1 (R_2 \vec{r} + \vec{\tau}_2) + \vec{\tau}_1$$

$$= \underline{\{R_1 R_2 \mid R_1 \vec{\tau}_2 + \vec{\tau}_1\} \cdot \vec{r}}$$

matrix rep.

$$\{g \mid \vec{\tau}\} = \begin{array}{c} \left(\begin{array}{c|c} 1 & 0 \\ \hline \vec{\tau} & R_g \end{array} \right) \end{array} \begin{array}{c} 1 \\ 3 \end{array} \qquad \vec{r} \to \begin{pmatrix} 1 \\ \vec{r} \end{pmatrix}$$

$$\{R_1 \mid \vec{\tau}_1\}\{R_2 \mid \vec{\tau}_2\} = \begin{pmatrix} 1 & 0 \\ \tau_1 & R_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \tau_2 & R_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ R_1 \tau_2 + \tau_1 & R_1 R_2 \end{pmatrix}$$

$$= \{R_1 R_2 \mid R_1 \vec{\tau}_2 + \vec{\tau}_1\}$$

4. $SU(2) \times SU(2) \to SO(4)$

$$M_x := x^\mu \tau_\mu \qquad \tau_\mu = \{\mathbb{1}_2, i\sigma^1, i\sigma^2, i\sigma^3\}$$

$$= x^0 \cdot \mathbb{1} + i \vec{x} \cdot \vec{\sigma}$$

$$= \begin{pmatrix} x^0 + i x^3 & i x^1 + x^2 \\ i x^1 - x^2 & x^0 - i x^3 \end{pmatrix}$$

$$= \{M \in M_2(\mathbb{C}) : M^* = \sigma_2 M \sigma_2\}$$

$\det M_x = |X|^2$: metric in $\mathbb{R}^4$.

Define left action $SU(2) \times SU(2)$:

$$M \longrightarrow u_1 M u_2^{-1}$$

We thus define a homomorphism

$$R : SU(2) \times SU(2) \longrightarrow SO(4)$$

$$\ker R \cong \mathbb{Z}_2 = \{(1,1), (-1,-1)\}$$

<u>Definition</u> (Orbits). Let X be a G-set

the orbit of G through a point $x \in X$. is

the set

$$O_G(x) := \{ g \cdot x \mid \forall g \in G \}$$

$$= \{ y \in X : \exists g, \text{ s.t. } y = g \cdot x \}$$

This defines an equivalence relation "$\sim$"

($x \sim x$; $x \sim y \Leftrightarrow y \sim x$; $x \sim y$. $y \sim z \Rightarrow x \sim z$)

$O_G(x)$ are equivalence classes ($[x]$) under

group action.

Distinct orbits of G partition X:

① $\forall x (\in X) \in O_G(x)$

② If $O_G(x_1) \cap O_G(x_2) \ni x \Rightarrow O_G(x_1) = O_G(x_2)$

$(x = g_1 \cdot x_1 = g_2 \cdot x_2 \qquad \underline{x_1} = g_1^{-1} \cdot g_2 \cdot \underline{x_2})$

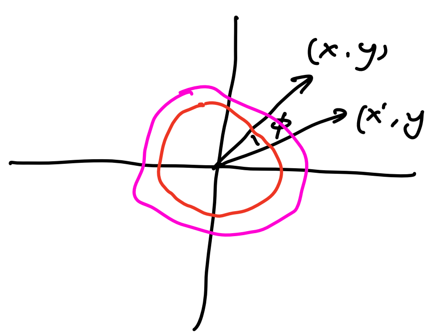
$\Rightarrow$ X is covered by disjoint orbits.

The set of orbits is denoted as X/G

Examples

1. $G = SO(2, R)$, on R^2

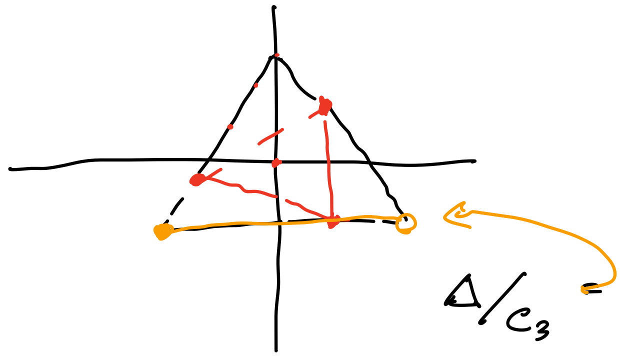
$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x\cos\phi - y\sin\phi \\ x\sin\phi + y\cos\phi \end{pmatrix}$$



$$R^2/SO(2) = [0, +\infty)$$

2. $G = C_3 = \{ R(0), R(2\pi/3), R(4\pi/3) \} \cong Z_3$
$\subset SO(2, R)$

X: eq lat. triangle



2b. space group $\{R|\tau\}$

set of orbits are "Wyckoff position".

3. $G = GL(n, k)$, $X = k^n$ n-dim vector space over k.

orbits: $\mathcal{O}_0 = \{\vec{0}\}$

$\mathcal{O}_* = \{\vec{x} \in k^n \mid \vec{x} \neq \vec{0}\}$

<u>Quantum mechanics</u>: "$\varphi \in \mathcal{H}$".

ray: $\varphi \sim c\varphi \quad (\forall c \in \mathbb{C})$

$\mathbb{C}^* = \mathbb{C} - \{0\}$ $\qquad \mathbb{C}^n - \{\vec{0}\}/\mathbb{C}^* = \mathbb{C}P^{n-1}$

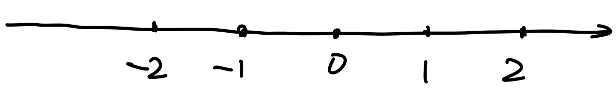
4. $G = \langle g \rangle \cong \mathbb{Z}$. $\qquad \mathbb{Z} = \langle 1 \rangle$

$\mathcal{O}_x = \{g^n \cdot x, \ n \in \mathbb{Z}\}$

$\mathbb{Z}$ on $\mathbb{R}$:

$n: x \longmapsto x + n$

$\mathbb{R}/\mathbb{Z} = [0, 1) \sim S^1$



5. $G = \mathbb{Z}_2 = \{e, \sigma\}$ on $\mathbb{R}^{n+1}$

$$\sigma \cdot (x^1, x^2, \dots x^{n+1})^T = (x^1, \dots x^p, -x^{p+1}, \dots, -x^{p+q})^T$$

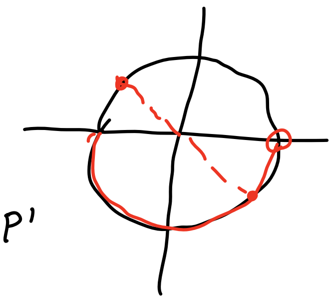
$(p+q = n+1)$

under this group action $\sum_i (x^i)^2 - 1 = 0$ is preserved

① $p = 0$, $q = n+1$

$\sigma \cdot \vec{x} = -\vec{x}$

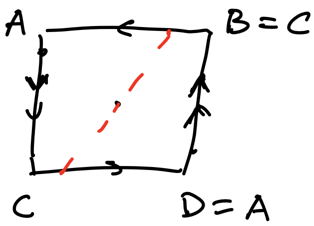
$S^1/\mathbb{Z}_2 =$  $= S^1 = \mathbb{R}P^1$

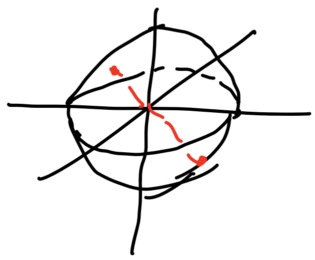


$\mathbb{R}P^1 = S^1$

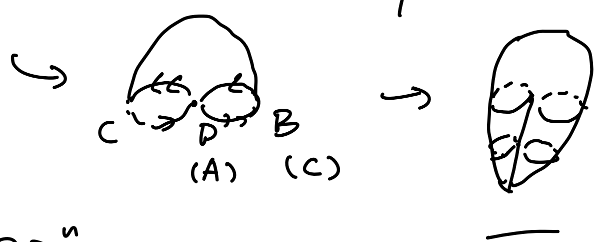
$\mathbb{R}P^n \neq S^n$

$(n \geq 2)$





$S^2/\mathbb{Z}_2 = \mathbb{R}P^2$



$S^n/\mathbb{Z}_2 = \mathbb{R}P^n$

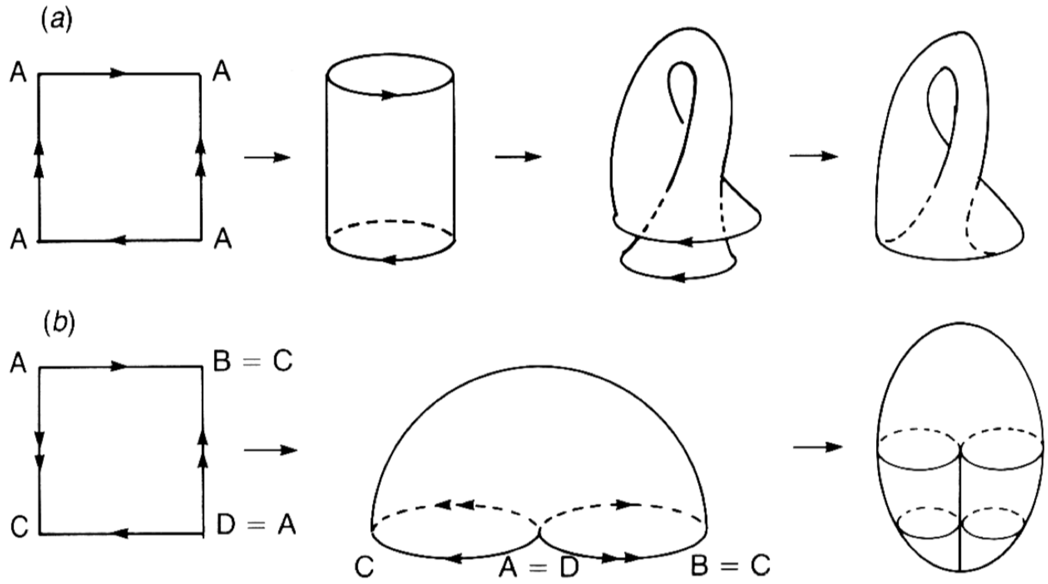


Figure 2.5. The Klein bottle (a) and the projective plane (b). ($\mathbb{R}P^2$)

Source: p73 of Nakahara, *Geometry, topology and physics*