Group theory { Groups

Group representation

group ⟺ symmetry ⟺ conservation

① special relativity. Minkowski space

$\mathbb{R}^{1,3} \rtimes O(1,3)$

general relativity. diffeomorphism invariance

② Lie group. $SU(2)$ $SO(3)$ $Sp(2n)$

↪ $\{ q_i, p_j \} = \delta_{ij}$

③ standard model

$\underbrace{U(1) \times SU(2)}_{\text{electro-weak}} \times \underbrace{SU(3)}_{\text{strong}}$

④ condensed matter

A. translational symmetry ($T^n = 1$) ⟺ momentum $\vec{k}$

B. rotations ⟷ point group (32)

A & B: 230 space groups (3D)

17 (2D)

2. Groups: Basic definitions & examples

Def: A group is a quartet $(G, \underline{m}, \underline{I}, e)$

1. G is a set.

2. $\underline{m}: G \times G \longrightarrow G$. "multiplication" map (closure)

3. $\underline{I}: G \longrightarrow G$ inversion map

4. $e \in G$. identity element.

They satisfy the following conditions:

1. (associativity).

$$\underline{m}(\underline{m}(g_1, g_2), g_3) = \underline{m}(g_1, \underline{m}(g_2, g_3))$$

$$\hookrightarrow (g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$$

Counter example: octonians (八元数)

2. (existence of identity) $\exists e$. s.t. $\forall g \in G$

$$e \cdot g = g \cdot e = g$$

3. (existence of inverse) $\forall g \in G.\ \exists\ \underline{I}(g) =: g^{-1} \in G.$

$$g \cdot g^{-1} = g^{-1} \cdot g = e$$

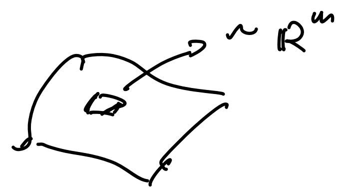
Remarks:

1. $e = 1 = 1_G = g_0 = 0$

2. related definitions.

$$\underbrace{\underbrace{\underbrace{\text{associativity}}_{\text{semigroup}},\ \exists e}_{\text{monoid}},\ \exists g^{-1}}_{\text{group}}$$

3. G is a manifold



Lie group. $\underline{m}, \underline{I}$ real analytic in local coordinates

4. $(G, \underline{m}, \underline{I}, e) =: G.$

Q :

a. "e" is unique ?

$e_1 = e_1 e_2 = e_2$

b. $\forall g \in G$. g^{-1} is unique (HW)

Examples :

1. $G = \mathbb{Z}, \mathbb{R}$. or $\mathbb{C}$.

$$\begin{cases} \underline{m}(a,b) := a+b & a,b \in G. \\ e : 0 \\ I : - \end{cases}$$

$\underline{m}(a,b) := ab$?

2. $G = \mathbb{R}^* := \mathbb{R} - \{0\}$

$\mathbb{C}^* := \mathbb{C} - \{0\}$

$\mathbb{Z}^* := \mathbb{Z} - \{0\}$?

Definition (subgroup)

(G, m, I, e) is a group. set $H \subset G$

m, I preserve H. i.e. $m: H \times H \to H$

$I: H \to H$

(H, m, I, e) is a subgroup of (G, m, I, e)

if $H \neq G$ "proper subgroup"

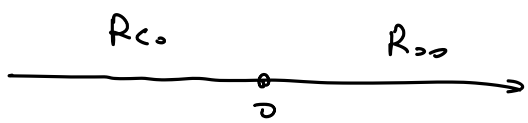
Q:

1 $\mathbb{Z} \subset \mathbb{R} \subset \mathbb{C}$. define subgroups

m. "$+$"

2. $\mathbb{Z}^* = \mathbb{Z} - \{0\} \subset \mathbb{R}^*$ is not a subgroup.

3. $\mathbb{R}^*$



$\underline{\underline{R_{>0}}}$

$R_{<0}$: $a \cdot (-1) := a$ $a \cdot \frac{1}{a} = -1$

4. subgroups H_1, H_2 of G

(a) is $H_1 \cap H_2$ a subgroup?

(b) is $H_1 \cup H_2$ a subgrop? (HW)

Definition (order of a group) $|G|$ is the cardinality of set G.

finite group if $|G| < \infty$ otherwise infinite group

Example. the group of Nth roots of unity

$$\mu_N = \{1, \omega, \dots, \omega^{N-1}\} = \{z \in \mathbb{C} \mid z^N = 1\}$$

$$\left(\omega = \exp\left(\frac{2\pi i}{N}\right)\right)$$

$$\omega^i \cdot \omega^j = e^{i \frac{2\pi}{N}(i+j \bmod N)} = \omega^k$$

$$k = i+j \bmod N.$$

$\simeq$ "$\mathbb{Z}_N$" m: $i+j \bmod N$

Definition (equivalence relation) "$\sim$" is a binary relation. s.t $\forall a, b, c \in$ a set X

(1) $a \sim a$ (reflexive)

(2) $a \sim b \Rightarrow b \sim a$ (symmetric)

(3) $a \sim b$, $b \sim c \Rightarrow a \sim c$ (transitive)

An equivalence class of X is a subset

$$[a] := \{x \in X \mid x \sim a\} \subset X$$

Example: residue classes modulo N.

$1 \le j \le N-1 \qquad [j] \,(= \bar{j}\,) = \{ n \in \mathbb{Z} \mid j = n \bmod N \}$

$$m(\bar{r}_1, \bar{r}_2) := \overline{r_1 + r_2}$$

$\mathbb{Z}_N$ or $\mathbb{Z}/N\mathbb{Z}$ " $/n$ "

$\mathbb{Z}_2 := \{0, 1\}$ $\qquad [0] = 0, 2, 4 \ldots$

$[1] = 1, 3 \ldots$

m : $i + j \bmod 2$

$\mathbb{Z}_2 := \{-1, 1\}$ $\qquad m$: $i \cdot j$

Definition. (direct product of groups) $G_1 \times G_2$

$$m_{G_1 \times G_2}((g_1, g_2), (g_1', g_2'))$$

$$:= (m_{G_1}(g_1, g_1'), \; m_{G_2}(g_2, g_2'))$$

$(g_1, g_1' \in G_1, \; g_2, g_2' \in G_2)$

Example. $G_1 = G_2 = \mathbb{Z}_2$

$\mathbb{Z}_2 \times \mathbb{Z}_2$: $\quad I = (1, 1)$ ✓

$a_1 = (-1, 1)$

$a_2 = (1, -1)$

$a_3 = (-1, -1)$

$I \cdot a_i = a_i \cdot I = a_i$; $\quad a_1 \cdot a_2 = (-1, -1) = a_3$

So far. $\underline{m}(a, b) = \underline{m}(b, a)$

<u>Definition</u> (Abelian & non-Abelian groups)

$$\begin{cases} \forall\, a, b \in G. \quad a \cdot b = b \cdot a. & \text{Abelian} \\ \exists\, a, b \in G. \quad s.t.\ a \cdot b \neq b \cdot a & \text{non-abelian} \end{cases}$$

hint ; $m(a, b) := a + b \qquad e = 0$

<u>Example</u> : (The general linear group)

$M_n(K)$: matrices defined on field K
$(n \times n)$ $\qquad (K = \mathbb{R}, \mathbb{C})$

$GL(n, K) := \{ A \in M_n(K) \mid A$ non-singular $\det A \neq 0 \}$

$AB \neq BA \qquad (n \geq 2)$

<u>Definition</u> (center of a group) $Z(G)$

$Z(G) := \{ z \in G \mid zg = gz, \ \forall g \in G \} \subset G$

↳ $Z(G)$ is an <u>Abelian</u> subgroup of G.

$Z(\underline{GL(n, K)}) = \{ \lambda \underline{\mathbb{1}_n}, \ \lambda \in K^* \} \qquad (*)$

Examples : standard matrix groups $\subset GL(n,K)$

1. special linear group

$$SL(n,K) = \{ A \in GL(n,K) \mid \det A = 1 \}$$

2. (special) orthogonal group

$$(\equiv A^T \cdot A)$$

$$O(n,K) := \{ A \in GL(n,K) \mid AA^T = 1 \} \qquad (\det A)^2 = 1, \quad \det A = \pm 1$$

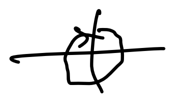
$$SO(n,K) := \{ A \in O(n,K) \mid \det A = 1 \}$$

3. (special) unitary group

$$(\det A)^* (\det A) = 1$$

$$U(n) := \{ A \in GL(n,\mathbb{C}) \mid AA^+ = 1 \} \qquad |\det A| = 1$$

$$SU(n) := \{ A \in U(n) \mid \det A = 1 \}$$



4. indefinite orthogonal group

$$O(p,q) := \{ A \in GL(p+q, \mathbb{R}) \mid A^T J_{p,q} A = J_{p,q} \}$$

$$J_{p,q} = \begin{pmatrix} -1_p & \\ & 1_q \end{pmatrix} \qquad J_{1,3} = \mathrm{diag}\{-1,1,1,1\}$$

Lorentz group $O(1,d)$ in $d+1$ space-time

5. symplectic group

$$Sp(2n,K) := \{ A \in GL(2n,K) \mid A^T J A = J \}$$

$$J = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} \qquad \left(J = J^* = -J^T = -J^{-1} \right)$$

Remarks.

1. $SO(2, \mathbb{R}) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \qquad a^2 + b^2 = 1$

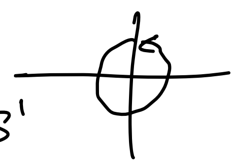
$$R(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} = e^{\phi J} \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$R(\phi_1)\, R(\phi_2) = R(\phi_1 + \phi_2)$$

2. $U(1):\quad z(\phi) = e^{i\phi} \qquad z(\phi_1)\, z(\phi_2) = z(\phi_1 + \phi_2)$

$(\mu_N \approx \mathbb{Z}_N)$

$U(1) \underset{\Delta}{\approx} SO(2)$ "$\approx$" S^1



3. $SU(2).\quad g = \begin{pmatrix} z & -w^* \\ w & z^* \end{pmatrix} \qquad |z|^2 + |w|^2 = 1$

$z = x_0 + i x_1 \qquad \hookrightarrow \sum_{i=0}^{3} x_i^2 = 1 \quad \sim S^3$

$z = x_2 + i x_3$

4. $SU(3)$ no simple geometric interpretation

"S^3-bundle over S^5"

5. $Sp(2n, K) \qquad A^T J A = J$

$\rightarrow (\det A)^2 = 1 \qquad \det A = \pm 1$

(math) $\Rightarrow \det A = 1$

$k = \mathbb{R}$ "Pfaffian" $Pf(\underbrace{A^T J A}_{\substack{\| \\ J}}) = \det(A) \cdot Pf(J)$

$\Rightarrow \det A = 1$

otherwise <u>Rim. arXiv 1505.04240</u>