Problem 23 (Haar measure of SU(2))[1]

Recall that an element of $SU(2)$ $g = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}$ can be expressed using Euler angles with $\alpha = e^{\mathrm{i}\frac{1}{2}(\phi+\psi)} \cos\theta/2$ and $\beta = \mathrm{i}e^{\mathrm{i}\frac{1}{2}(\phi-\psi)} \sin\theta/2$

(a) Verify that $[dg] = \frac{1}{16\pi^2} d\psi d\phi \sin\theta d\theta$ is the normalized Haar measure.

(b) Show that

$$\int_{SU(2)} dg g_{\alpha\beta} = 0$$

$$\int_{SU(2)} dg g_{\alpha\beta} g_{\gamma\delta} = \frac{1}{2} \epsilon_{\alpha\gamma} \epsilon_{\beta\delta}.$$

ϵ here is the Levi-Civita / totally antisymmetric tensor. In 2D, $\epsilon = \mathrm{i}\sigma_2$.

(c) Using left and right-invariance show that

$$\int_{SU(2)} dg g_{\alpha_1\beta_1} \cdots g_{\alpha_n\beta_n}$$

can be nonzero if n is even and half of the α's (β's) are 1.

Problem 24

Find three different irreps for S_3. (*Hint:* two are one dimensional and one two dimensional.)

[1] pp.236-237 of [GM]