Problem 01 (from the lecture)

Given a group G, show that

(1) the identity element e is unique, and

(2) $\forall a \in G$, a^{-1} is unique.

Problem 02 (from the lecture)

Suppose H_1 and H_2 are two subgroups of a group G.

(1) Show that $H_1 \cap H_2$ is also a subgroup of G.

(2) Under what condition is $H_1 \cup H_2$ a subgroup of G?

Problem 03

Let G be a group, s.t. $\forall g \in G$, $g^2 = e$. Show that G is abelian.

Problem 04

Prove the following theorem: A subset H of a group G is a subgroup if and only if $e \in H$ and $h_1, h_2 \in H$ imply $h_1 h_2^{-1} \in H$.

Problem 05

Show that a general element $g \in SU(2)$ is of the form

$$g = \begin{pmatrix} z & -\omega^* \\ \omega & z^* \end{pmatrix}$$

with $(z, \omega) \in \mathbb{C}$ and $|z|^2 + |\omega|^2 = 1$.

Problem 06 (p. 24 of [GM] [1])

Let $\mathbf{q} = \{q_1, q_2, \ldots, q_n\}^T$ and $\mathbf{p} = \{p_1, p_2, \ldots, p_n\}^T$ be coordinates and momenta for a classical mechanical system. The Poisson bracket of two functions $f(\mathbf{q}, \mathbf{p})$ and $g(\mathbf{q}, \mathbf{p})$ is defined to be

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right).$$

Show that

[1] Greg Moore notes, Chapter 01, http://www.physics.rutgers.edu/ gmoore/618Spring2023/GTLect1-AbstractGroupTheory-2023.pdf

(1) $\{q_i, q_j\} = \{p_i, p_j\} = 0$ and $\{q_i, p_j\} = \delta_{ij}$

(2) A new set of coordinates and momenta $\mathbf{Q} = \{Q_1, Q_2, \ldots, Q_n\}^T$, $\mathbf{P} = \{P_1, P_2, \ldots, P_n\}^T$ defined as

$$\begin{pmatrix} \mathbf{Q} \\ \mathbf{P} \end{pmatrix} = A \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix}$$

via a constant $2n \times 2n$ matrix A still satisfy the relation in (1), if and only if $A \in Sp(2n, \mathbb{R})$.