P 2-1.

(1)

G	$\hat{G}$
$(\mathbb{R}, +)$	$(\mathbb{R}, +)$
$(U(1), \times)$	$(\mathbb{Z}, +)$
$(\mathbb{Z}, +)$	$(U(1), \times)$
$(\mathbb{Z}_N, +)$	$(\mathbb{Z}_N, +)$

see lecture.

(2) $\hat{f}(\chi) = \int_G f(g)\, \chi(g)\, dg$

① $x \in \mathbb{R}$. $\chi_k(x) = e^{ikx}$, $(k \in \mathbb{R})$

$$\hat{f}(k) = \int dx\, e^{ikx} f(x)$$

$$f(x) = c \cdot \int dk\, e^{-ikx} \hat{f}(k)$$

$$f(x) = c \int dk\, e^{-ikx} \int dx'\, e^{ikx'} f(x')$$

$$= c \cdot 2\pi \int dx'\, \delta(x'-x) f(x')$$

$$= c \cdot 2\pi \cdot f(x)$$

$c = \frac{1}{2\pi}$. and two transforms are inverses of each other.

②/③: $e^{i\theta} \in U(1)$ $\quad \chi_n(e^{i\theta}) = e^{in\theta}$ $\quad \theta \in [0, 2\pi)$

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} e^{in\theta} f(\theta)\, d\theta \qquad \left(\frac{1}{2\pi}\int_0^{2\pi} d\theta = 1\right)$$

$$f(\theta) = \sum_{-\infty}^{\infty} e^{-in\theta} \cdot \hat{f}(n)$$

$$f(\theta) = \sum_{-\infty}^{\infty} e^{-in\theta} \cdot \frac{1}{2\pi}\int_0^{2\pi} e^{in\theta'} f(\theta')\, d\theta'$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_0^{2\pi} e^{in(\theta'-\theta)} f(\theta')\, d\theta'$$

$$= \sum_{l=-\infty}^{\infty}\int_0^{2\pi} \delta[(\theta'-\theta) - 2l\pi]\, f(\theta')\, d\theta'$$

$$= f(\theta)$$

④: $m \in \mathbb{Z}_N$ $\quad \chi_n(m) = \omega^{mn}$ $\quad \omega = e^{i\frac{2\pi}{N}}$

$$\hat{f}(n) = \frac{1}{N}\sum_{m=0}^{N-1} e^{i\frac{2\pi mn}{N}} f(m)$$

$$f(m) = \sum_{n=0}^{N-1} e^{-i\frac{2\pi mn}{N}} \hat{f}(n)$$

$$f(m) = \frac{1}{N}\sum_{n,l} e^{-i\frac{2\pi mn}{N}}\, e^{i\frac{2\pi ln}{N}} f(l)$$

$$= \frac{1}{N}\sum_{l}\sum_{n} e^{i\frac{2\pi n}{N}(l-m)} f(l)$$

$$= \sum_{l} \delta_{lm} f(l)$$

$$= f(l)$$