

Problem 19 (Haar measure of $SU(2)$)¹

Recall that an element of $SU(2)$ $g = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}$ can be expressed using Euler angles with $\alpha = e^{i\frac{1}{2}(\phi+\psi)} \cos \theta/2$ and $\beta = ie^{i\frac{1}{2}(\phi-\psi)} \sin \theta/2$

(a) Verify that $[dg] = \frac{1}{16\pi^2} d\psi d\phi \sin \theta d\theta$ is the normalized Haar measure.

(b) Show that

$$\int_{SU(2)} dg g_{\alpha\beta} = 0$$

$$\int_{SU(2)} dg g_{\alpha\beta} g_{\gamma\delta} = \frac{1}{2} \epsilon_{\alpha\gamma} \epsilon_{\beta\delta}.$$

ϵ here is the Levi-Civita / totally antisymmetric tensor. In 2D, $\epsilon = i\sigma_2$.

(c) Using left and right-invariance show that

$$\int_{SU(2)} dg g_{\alpha_1\beta_1} \cdots g_{\alpha_n\beta_n}$$

can be nonzero if n is even and half of the α 's (β 's) are 1.

Problem 20

Find three different irreps for S_3 . (*Hint*: two are one dimensional and one two dimensional.)

¹pp.192-193 of [GM]