

**Problem 16 (Real representation)** <sup>1</sup>

Let  $V$  be a complex vector space. The complex conjugate representation sends  $g \rightarrow \bar{T}(g) \in GL(\bar{V})$ . If  $\{v_i\}$  is an ordered basis for  $V$  then there is a canonical ordered basis  $\{\bar{v}_i\}$  for  $\bar{V}$ .

A *real representation* is one where  $(\bar{T}, \bar{V})$  is equivalent to  $(T, V)$ . Show that for a real representation  $T$ , there exists  $S \in GL(n, \mathbb{C})$  s.t.  $\forall g \in G$ ,

$$M_T^*(g) = SM_T(g)S^{-1}$$

**Problem 17** <sup>2</sup>

Let  $V = \mathbb{C}^n$  and  $W = \mathbb{C}^m$  be representation spaces for  $G$ . The vector space of linear transformations from  $V$  to  $W$ , denoted  $Hom(V, W)$  is also a representation, via the identification  $Hom(V, W) = V^* \otimes W$ . Define

$$(\tilde{T}(g) \cdot \phi)(v) = T_W(g) \cdot \phi(T_V(g^{-1})v)$$

for  $\phi \in Hom(V, W)$  and  $v \in V$ .

- (1) Verify that it is indeed a representation.
- (2) Show that the dual representation is a special case of this representation.
- (3) Equip  $V$  with basis  $\{e_i\}$  and  $W$  with basis  $\{e_a\}$ . Identify  $Hom(V, W) \cong Mat_{m \times n}(\mathbb{C})$ , show that

$$\tilde{T}(g)e_{ai} = M_{ba}(g)(M(g)^{tr, -1})_{ki}e_{bk},$$

where  $e_{ij}$  are the unit matrices (1 at  $\{i, j\}$ , 0 otherwise).

**Problem 18**

For a representation  $(T, V)$  of a *finite group*  $G$  on an inner product space  $V$ . Define an inner product s.t.

$$\langle T(g)v, T(g)w \rangle = \langle v, w \rangle, \quad \forall v, w \in V, \forall g \in G.$$

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<sup>1</sup>p.183 of [GM]

<sup>2</sup>pp.182-183 of [GM]