P19. Haar measure

(a) see lecture notes.

<u>Option 1</u>: use left- & right-invariance

(b) $\phi_{\alpha\beta} \equiv \int dg\, g_{\alpha\beta} = \int dg\, (g_0 g)_{\alpha\beta} = g_{0\alpha\gamma} \int dg\, g_{\gamma\beta}$

$$g_0 \begin{pmatrix} \phi_{0\beta} \\ \phi_{1\beta} \end{pmatrix} = \begin{pmatrix} \phi_{0\beta} \\ \phi_{1\beta} \end{pmatrix} \quad (\forall g_0 \in SU(2))$$

choose $g_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\quad \phi_{0\beta} = \pm \phi_{1\beta} = 0$

$$\Rightarrow \underline{\int dg\, g_{\alpha\beta} = 0 \qquad \forall \alpha, \beta \in \{0, 1\}}$$

$$(A^{\beta\delta})_{\alpha\gamma} = \int dg\, g_{\alpha\beta} g_{\gamma\delta} = \int dg\, (g_0 g)_{\alpha\beta} (g_0 g)_{\gamma\delta}$$

$$= (g_0)_{\alpha s} \int dg\, g_{s\beta} g_{t\delta} (g_0)_{\gamma t}$$

$$\Rightarrow A^{\beta\delta} = g_0 \cdot A^{\beta\delta} \cdot g_0^T \quad (\forall g_0 \in SU(2))$$

$A^{\beta\delta} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ take $g_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ & $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

$$\Rightarrow A^{\beta\delta} = c_{\beta\delta} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Similarly. $A^{\alpha\gamma} = c_{\alpha\gamma} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ by right-invariance

$$\Rightarrow A_{\alpha\gamma,\beta\delta} = c_{\beta\delta}\, \epsilon_{\alpha\gamma} = c_{\alpha\gamma} \cdot \epsilon_{\beta\delta}$$

$$\Rightarrow A_{\alpha\gamma,\beta\delta} = \underline{c \cdot \epsilon_{\alpha\gamma} \epsilon_{\beta\delta}} \qquad c = \tfrac{1}{2} \text{ by explicit calculation.}$$

(c) $I = \int dg\, g_{\alpha_1\beta_1} \cdots g_{\alpha_n\beta_n}$

$$= \int dg\, (g_0 g)_{\alpha_1\beta_1} \cdots (g_0 g)_{\alpha_n\beta_n} \qquad (*)$$

① $g_0 = -\mathbb{1}$. $(*) \Rightarrow I = (-1)^n I \Rightarrow I = 0$ for <u>odd $n$</u>.

② $n$ even: $g_0 = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$ $(g_0)_{\alpha\beta} = \delta_{\alpha\beta}\, e^{(-1)^\alpha i\theta}$

$(g_0 g)_{\alpha\beta} = e^{(-1)^\alpha i\theta} g_{\alpha\beta}$

$(*) \Rightarrow I = e^{i\theta \sum (-1)^{\alpha_i}} I \overset{I \neq 0}{\Rightarrow} \sum (-1)^{\alpha_i} = 0 \Rightarrow$ half $\alpha_i$: 1
half $\alpha_i$: 2

similarly, by right-invariance, half $\beta_i$: 1.

<u>Option 2:</u> explicit calculation

(b) ① $\int_{SU(2)} dg\, g_{ab} \propto \int_0^{4\pi} d\varphi\, e^{\pm i\varphi/2} = 0$

② $\int_{SU(2)} dg\, g_{ab} g_{cd} = \frac{1}{2} \epsilon_{\alpha\gamma} \epsilon_{\beta\delta}$

show by explicit computation
zero terms contain phases $e^{\pm i\phi}$ $(\phi \in [0, 2\pi))$
or $e^{\pm i\frac{\varphi}{2}}$ $(\varphi \in [0, 4\pi))$.

(c) $\int_{SU(2)} dg\, g_{\alpha_1\beta_1} \cdots g_{\alpha_n\beta_n}$ to be nonzero.

① n odd : must contain factor $e^{\pm i\frac{\phi}{2}} \Rightarrow 0$

② n even: each $\alpha$ should be paired with $\alpha^*$

i.e. $g_{11}$ paired with $g_{22}$

similar. $g_{12}$ paired with $g_{21}$.

$\Rightarrow$ half indices are 1 and the other half 2

P20. three irreps of $S_3$:

① trivial: $\rho(\phi) = 1 \quad \forall \phi \in S_3$

② sign-rep: $\rho(\phi) = \mathrm{sgn}(\phi)$.

③ $S_3 \cong D_3$ . 2x2 rotation/reflection matrices

see lecture notes.