P10.

$\phi_1(u) = u. \quad \phi_2(u) = u^*$

(1) $u = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \quad |\alpha|^2 + |\beta|^2 = 1$

$\mathrm{tr}\, u = \mathrm{tr}\, u^*$

$(i\sigma^2)\, u\, (i\sigma^2)^{-1} = u^*$

(2) eigenvalues of unitaries $|\lambda| = 1$

$\mathrm{tr}\, u = e^{i\theta_1} + e^{i\theta_2}$

$\mathrm{tr}\, u^* = e^{-i\theta_1} + e^{-i\theta_2}$

in general not the same.

(3) $\mathrm{tr}\, u = \sum_{i=1}^{n} e^{i\theta_i} \quad \& \left(\prod e^{i\theta_i} = 1 \Leftrightarrow \sum \theta_i = 0 \mod 2\pi \right)$

$\mathrm{tr}\, u^* = \sum_{i=1}^{n} e^{-i\theta_i}$

in general not the same.

P 11.

(1) Consider left cosets H & gH $\quad (g \notin H)$

right cosets H & Hg

both partition G. $\quad (\because [G:H]=2)$

$\Rightarrow gH = Hg$

(2) $G/Z(G) = \langle g Z(G) \rangle$ cyclic

$\forall a, b \in G$. they are in some cosets.

WLOG. $a = g^m z_1 \in g^m Z(G)$ $\quad b = g^n z_2 \in g^n Z(G)$

$ab = g^m z_1 g^n z_2 = g^{m+n} z_1 z_2 = g^{n+m} z_2 z_1$

$= g^n z_2 g^m z_1 = ba$

$\Rightarrow G$ abelian.

P12. (1) $A \in GL$. $B \in SL$

$\det(BAB^{-1}) = \det(A) = 1$

$BAB^{-1} \in SL$

(2) Verify specifically. or use the fact that $[S_n : A_n] = 2$ and the statement of P11 (1).

PB. (1) $[g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1}$

$$g[a,b]g^{-1} = gaba^{-1}b^{-1}g^{-1} = (gag^{-1})(gbg^{-1})(ga^{-1}g^{-1})(gb^{-1}g^{-1}),$$

$$= [gag^{-1}, gbg^{-1}] \in [G,G]$$

Then any products of the generators $\tau_i = [a_i, b_i] \in [G,G]$

$$g(\Pi\, \tau_i)g^{-1} = \Pi\, g\tau_i g^{-1} \in [G,G]$$

$$\Rightarrow g[G,G]g^{-1} = [G,G] \quad \forall g \in G.$$

(2) $H \triangleleft G$, $(G/H \text{ abelian}) \Leftrightarrow [G,G] \subset H$.

(a) $\Rightarrow$: $(aH)\cdot(bH) = abH = (bH)\cdot(aH) = baH \quad (a, b \in G)$

$\Rightarrow \quad abh_1 = bah_2 \qquad (h_i \in H)$

$\Rightarrow a^{-1}b^{-1}ab = h_2 \cdot h_1^{-1} \in H$

$\Rightarrow [a^{-1}, b^{-1}] \in H$

(b) $\Leftarrow$: $[a, b] \in H$

$\Rightarrow aba^{-1}b^{-1} = h \in H$

$\Rightarrow a^{-1}b^{-1} = b^{-1}a^{-1}h$

$\Rightarrow a^{-1}b^{-1}H = b^{-1}a^{-1}H$

$\Rightarrow (a^{-1}H)\cdot(b^{-1}H) = (b^{-1}H)\cdot(a^{-1}H).$

$\Rightarrow G/H$ abelian.