P 4. Canonical transformations

(1) trivial.

(2) Def in lecture

$$Sp(2n, K) := \{ A \in GL(2n, K) \mid A^T J A = J \}$$

equiv. $AJA^T = J$

$$J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \qquad J = J^* = -J^T = -J^{-1}$$

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -A_{12} & A_{11} \\ -A_{22} & A_{21} \end{pmatrix} \begin{pmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \end{pmatrix}$$

$$= \begin{pmatrix} A_{11}A_{12}^T - A_{12}A_{11}^T & A_{11}A_{22}^T - A_{12}A_{21}^T \\ A_{21}A_{12}^T - A_{22}A_{11}^T & A_{21}A_{22}^T - A_{22}A_{21}^T \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\boxed{\begin{aligned} \Rightarrow \ & (A_{11}A_{12}^T - A_{12}A_{11}^T)_{ij} = 0 \qquad \forall\, i\,j \in [1, n] \\ & (A_{11}A_{22}^T - A_{12}A_{21}^T)_{ij} = \delta_{ij} \end{aligned}}$$

$$\begin{pmatrix} \vec{Q} \\ \vec{P} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \vec{q} \\ \vec{p} \end{pmatrix} = \begin{pmatrix} A_{11}\vec{q} + A_{12}\vec{p} \\ A_{21}\vec{q} + A_{22}\vec{p} \end{pmatrix}$$

$$Q_i = \sum_{j=1}^{n} (A_{11})_{ij}\, q_j + \sum (A_{12})_{ij}\, p_j$$

$$P_i = \sum_{j=1}^{n} (A_{21})_{ij}\, q_j + \sum (A_{22})_{ij}\, p_j$$

$$\frac{\partial Q_i}{\partial q_l} = (A_{11})_{il} \qquad \frac{\partial Q_i}{\partial p_l} = (A_{12})_{il}$$

$$\frac{\partial P_i}{\partial q_l} = (A_{21})_{il} \qquad \frac{\partial P_i}{\partial p_l} = (A_{22})_{il}$$

$$\{Q_i, Q_j\} = \sum_l \left(\frac{\partial Q_i}{\partial q_l}\frac{\partial Q_j}{\partial p_l} - \frac{\partial Q_i}{\partial p_l}\frac{\partial Q_j}{\partial q_l} \right) = \sum_l \left[(A_{11})_{il}(A_{12})_{jl} - (A_{12})_{il}(A_{11})_{jl} \right]$$

$$= (A_{11}A_{12}^T - A_{12}A_{11}^T)_{ij} = 0 \quad \checkmark$$

$\{P_i, P_j\}$ is similar.

$$\{Q_i, P_j\} = \sum_l \left(\frac{\partial Q_i}{\partial q_l}\frac{\partial P_j}{\partial p_l} - \frac{\partial Q_i}{\partial p_l}\frac{\partial P_j}{\partial q_l} \right)$$

$$= \sum_l \left[(A_{11})_{il}(A_{22})_{jl} - (A_{12})_{il}(A_{21})_{jl} \right]$$

$$= (A_{11}A_{22}^T - A_{12}A_{21}^T)_{ij} = \delta_{ij} \quad \checkmark$$

P5. Quaternion $\to V$.

There are many homomorphisms.

One example:

$Q = \{\pm 1, \pm i, \pm j, \pm k\}$ $\quad V = \{1, a, b, ab\}$

$\varphi: Q \to V$

define $\varphi(i) = a$. $\varphi(j) = b$ $\quad \varphi(k) = \varphi(ij) = \varphi(i)\varphi(j) = ab$

$\varphi(-1) = \varphi(i)\varphi(i) = a^2 = 1$

$\varphi(-i) = \varphi(i)\varphi(-1) = a$

$\varphi(-j) = b$. $\quad \varphi(-k) = ab$

$\ker\varphi = \{\pm 1\} \cong \mathbb{Z}_2$

$\operatorname{im}\varphi = V$

P6.

$$
\begin{tikzcd}
\mathbb{Z}_N \arrow[r, "m_{k_1}"] \arrow[d, "\varphi"] & \mathbb{Z}_N \arrow[d, "\varphi"] \\
\mu_N \arrow[r, "\rho_{k_2}"] & \mu_N
\end{tikzcd}
$$

commutes iff $k_1 = k_2 \bmod N$

$\Leftarrow$ trivial.

$\Rightarrow$ if $k_1 \neq k_2 \bmod N$

$\Lsh$: $\rho_{k_2}(\varphi(\bar{i})) = \rho_{k_2}(\omega^{i+nN}) = \omega^{ik_2 \bmod N}$

$\urcorner$: $\varphi(m_{k_1}(\bar{i})) = \varphi(\overline{k_1 i}) = \omega^{ik_1 \bmod N}$

$\forall i, \; ik_1 = ik_2 \bmod N$

$k_1 = k_2 \bmod N.$